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The Cohen–Macaulay Property and F-Rationality in Certain Rings of Invariants

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Consider the action of a group $G \leq S_n$ that permutes the n variables in a polynomial ring $k[X_1, \dots, X_n]$ over a field k. Two related properties, the Cohen–Macaulay property and F-rationality, are studied in the ring of invariants, and the following results are obtained. (1) The invariant ring $k[X_1, \dots, X_n]^{C_n}$ produced by cyclic permutation of the variables is shown *not* to be Cohen–Macaulay in characteristics dividing n for $n > 4$. This completes the analysis of the characteristics in which this invariant ring is Cohen–Macaulay. (2) The non-F-rational locus of $k[X_1, \dots, X_n]^{A_n}$ is found to have positive dimension for certain n and k, although this ring possesses many of the properties of F-rational rings.

Any finite group G can be made to act on a polynomial ring in a natural and nontrivial way: Embed G as a subgroup in a permutation group S_n for some n, fix a field k, and let G act on $k[X_1, \dots, X_n]$ according to the k-algebra action determined by $\sigma(X_i) := X_{\sigma(i)}$, $\sigma \in G$. Such actions ("variable permuting actions") seem innocuous enough to produce good properties in the ring of invariants $k[X_1, \dots, X_n]^G$. Of course, this intuition holds true to a certain extent, for the invariant ring enjoys certain basic properties, including evidence that all ideals might be *tightly closed*. Specifically, this ring is well known to be a normal graded F-pure domain finitely generated over k and often has a negative a-invariant. (See Sections 1 and 2.)

Notably absent from this tight closure evidence is the Cohen–Macaulay property, which is known to be present in the ring $k[X_1, \dots, X_n]^G$ for

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some combinations of n, $G \leq S_n$, and k and to be absent for others. (See Section 3.) Consequently, it is natural to study two related issues:

QUESTION 1. *When is* $k[X_1, \dots, X_n]^G$ *Cohen–Macaulay? That is, for which choices of* n*, an embedding* $G \leq S_n$*, and* k *is the ring Cohen–Macaulay?*

QUESTION 2. *When are its ideals tightly closed? Or a less stringent question: When is* $k[X_1, \dots, X_n]^G$ *F-rational?*

These are the questions we study in this paper. Some remarks are in order at this point. Namely, there are some important initial remarks to make and we should briefly summarize prior results and results obtained in this paper.

Initial Remarks on Questions 1 and 2

Fixing n and $G \leq S_n$, both problems are quickly solved for most k, due to an important result of [HH1]: In all but the finitely many characteristics dividing $|G|$, the invariant ring $k[X_1, \dots, X_n]^G$ is well known to split off from the regular ring $k[X_1, \dots, X_n]$. The result of [HH1] then guarantees that such $k[X_1, \dots, X_n]^G$ are F-rational (hence also Cohen–Macaulay). *Thus, we only need to answer Questions* 1 *and* 2 *when* char(k) *divides* $|G|$.

Due to this reduction and our expectation that the salient feature of the field k is its characteristic, we like to think of the problems phrased as follows: Fixing n and $G \leq S_n$, we have a family of rings $k[X_1, \dots, X_n]^G$, parametrized by the characteristic, all but finitely many of which are Cohen–Macaulay and F-rational. Now the question is: What happens in these finitely many characteristics (those dividing $|G|$)?

We note that the result of [HR1], asserting that the action of a linearly reductive group on a regular ring produces a Cohen–Macaulay invariant ring, does not apply to Question 1 because a finite group G is not linearly reductive in characteristics dividing its order.

Prior Results for Questions 1 and 2

Given how many nice properties the rings $k[X_1, \dots, X_n]^G$ have and the fundamental nature of the questions, it is surprising how little is known in characteristics dividing $|G|$.

Prior results for Question 1 in characteristics dividing $|G|$ only concern the groups $G = S_n, A_n$ (known classically) and some of the cyclic groups [Be, FG]. (In each case the embedding $G \leq S_n$ is the usual one.)

There are essentially no prior results on Question 2 in characteristics dividing $|G|$. One only had answers in trivial cases, such as when $k[X_1, \dots, X_n]^G$ is regular (and hence F-rational), as is the case for $G = S_n$, or not Cohen–Macaulay (hence not F-rational), as was known for the cyclic groups mentioned above.

See Section 3 for references and more detailed explanations of these results.

Results in This Paper

We answer Question 1 when G is cyclic and the embedding $G \leq S_n$ is the natural one (Theorem 7.2). (That is, embed $G = C_n$ in S_n by identifying a generator of G with the n-cycle $(12 \cdots n)$.) That is, we consider the k-algebra action of the cyclic group $C_n = \langle \sigma \rangle$ of order n on the polynomial ring $k[X_1, \dots, X_n]$ determined by

$$\begin{aligned} \sigma(X_1) &= X_2 \\ \sigma(X_2) &= X_3 \\ &\vdots \\ \sigma(X_{n-1}) &= X_n \\ \sigma(X_n) &= X_1. \end{aligned}$$

We show in 7.2 that the invariant ring $k[X_1, \dots, X_n]^{C_n}$ is Cohen–Macaulay if and only if $n \leq 3$ or char(k) is relatively prime to n.

A result (Theorem 4.1) that we obtain en route to 7.2 might be useful in answering Question 1 for any embedding $G \leq S_n$ of a finite abelian group G. This result concerns the étale locus of $k[X_1, \dots, X_n]^G \subseteq k[X_1, \dots, X_n]$. We explain how it might be used in a more general attack on Question 1 at the end of Section 7.

Theorem 7.2 extends naturally to certain embeddings $G \leq S_n$ of products of cyclic groups (Theorem 7.3). Thus, we answer Question 1 for *certain* embeddings of *any* finite abelian group G.

We answer Question 2 when $G = A_n$ (the alternating group) for certain values of char(k) dividing $|A_n| = n!/2$. (The embedding $A_n \leq S_n$ is taken to be the natural one.) Namely, we show that $k[X_1, \dots, X_n]^{A_n}$ is not F-rational if $n, p = \text{char}(k) \geq 3$ and $n \equiv 0$ or $1 \bmod p$ (Theorem 12.2), and, furthermore, that its non-F-rational locus has positive dimension (Theorem 12.3). In other words, these rings do not just fail to be F-rational at isolated points in their spectra. The results 12.2 and 12.3 are somewhat surprising because the rings $k[X_1, \dots, X_n]^{A_n}$ have the major characteristics of graded F-rational rings. (They are normal Cohen–Macaulay F-injective domains with negative a-invariants: See Sections 1 and 2.)

An interesting theme in our results is that the examined ring $k[X_1, \dots, X_n]^G$ fails to maintain, in at least some of the characteristics dividing $|G|$, a good property (Cohen–Macaulay or F-rationality) that it possesses in the other characteristics. (This is not a universal phenomenon: See Example 1 in Section 3.)

The rings in this paper are tacitly assumed to be Noetherian.

1. BACKGROUND: R^G AND ITS FREE LOCUS

This section contains background material concerning the ring R^G and the locus of primes of R^G on whose fibers G acts *freely* (i.e., the "free" locus in Spec R^G). We state this material in a fairly general setting that yields an efficient presentation and suits our purposes for later sections. Namely, R will denote a domain finitely generated over a field k of characteristic $p \geq 0$, and G will be a finite group acting on R by k-algebra automorphisms.

A classical result of E. Noether asserts that R^G is finitely generated over k. The invariant ring is also easily seen to be a normal domain when R is. (The G-action on R extends to the fraction field E of R, and E^G is the fraction field of R^G, so $R^G = E^G \cap R$ is an intersection of normal domains.) Note that these properties hold in any characteristic.

A well-known characteristic-sensitive property of R^G is that R^G is a direct summand of R as an R^G-module *when the characteristic does not divide* $|G|$. (One easily checks that the map $R \to R^G$ that sends

$$f \mapsto \frac{1}{|G|} \sum_{\sigma \in G} \sigma(f)$$

is R^G-linear and retracts the inclusion $R^G \subseteq R$.) In particular, R^G is a direct summand of R in characteristic zero. In the next section, we will see how this splitting property applied *when R is regular* forces R^G to be Cohen–Macaulay and F-rational in such characteristics.

When one is wondering about depths of ideals in R^G (as we will when considering whether R^G is Cohen–Macaulay), it is useful to have the following spectral sequences relating group and sheaf cohomology from [Gr].

1.1. PROPOSITION. *Let R be a finitely generated k-algebra and let G be a finite group acting on R by k-algebra automorphisms. Fix an ideal $I \subseteq R^G$. Consider the spectra $X :=$ Spec R, $Y :=$ Spec R^G and their open sets $\dot{X} := X \setminus V(IR)$, $\dot{Y} := Y \setminus V(I)$. There are two spectral sequences with the same abutment whose E_2^{pq} terms are given by*

$$\mathbf{I}: H^q\big(G, H^p\big(\dot{X}, \mathscr{O}_X\big)\big)$$

$$\mathbf{II}: H^p\big(\dot{Y}, \mathbf{H}^q(G, \mathscr{O}_X)\big),$$

where $\mathbf{H}^q(G, \mathscr{O}_X)$ is the sheaf associated to the presheaf on $\dot{Y}$ that sends an open set U to $H^q(G, H^0(V, \mathscr{O}_X))$, where V is the inverse image of U under $\dot{X} \to \dot{Y}$.

Proof. This is in [Gr]. These spectral sequences arise as the iterated homologies of the double complex

$$\operatorname{Hom}_{k[G]}(P_q, C^p),$$

where $P. \to 0$ is a projective resolution of k over $k[G]$ and $C^\cdot$ is the Cech complex of $\mathcal{O}_X$ with the open cover $\{D(x_i)\}_i$, where the x_i are a finite set of generators of I.

We now shift our focus to the fibers of $R^G \subseteq R$. In the remainder of this section, we discuss two related notions for the induced action of G on a fiber to be *free*. This action is described by $\sigma.Q := \sigma(Q)$ (for $Q \in \operatorname{Spec} R$ and $\sigma \in G$). Due to the incomparability of primes in a fiber (recalling that $R^G \subseteq R$ is module-finite), the action on any particular fiber is transitive. Thus, there are at most $|G|$ elements in a fiber, and G acts freely on a fiber, in the sense that every prime in it has a trivial stabilizer, precisely when that fiber has exactly $|G|$ elements. This is one notion of a free action on a fiber, but we would rather use the following related notion.

1.2. DEFINITION. We say that G acts freely over $P \in \operatorname{Spec} R^G$ (or acts freely on the fiber over P) if $R_P^G \subseteq R_P$ is étale.

A property of such primes key to our results in Part I is the following normal basis theorem from [Fo]. It guarantees a normal basis for $R_m^G \to R_m$ at a maximal ideal $m \subseteq R^G$ over which the action is free.

1.3. PROPOSITION. *Let A be a semilocal domain and let G be a finite group of automorphisms of A acting transitively on the maximal ideals of A. If A is a finite étale extension of A^G, then there exists an $f \in A$ such that $\{\sigma(f): \sigma \in G\}$ is a free basis of A over A^G.*

Proof. [Fo, Lemma 3].

We end this section by addressing two natural questions about our "free locus" (i.e., the locus in Spec R^G over which the action of G is free): the openness of this locus, and its relationship to the less subtle notion of freeness (that the fiber in question have $|G|$ elements). For this, we use the well-known fact that R is module-finite over R^G of torsion-free rank $|G|$. (R is easily seen to be algebra-finite and integral over R^G. The fraction field E of R is isomorphic to $R \otimes_{R^G} L$, where L is the fraction field of R^G. The induced action of G on E produces $L = E^G$ as invariants, so $\operatorname{rk}_{R^G} R = \dim_{E^G} E = |G|$.)

1.4. PROPOSITION. *Let R be a domain finitely generated over a field k, and suppose a finite group G acts on R by k-algebra automorphisms. Define*

$$U := \{P \in \operatorname{Spec} R^G : R_P^G \to R_P \text{ is étale}\}$$

$$V := \{P \in \operatorname{Spec} R^G : \text{the fiber over } P \text{ has } |G| \text{ elements}\}.$$

Then

(a) *U is Zariski open in* Spec R^G.

(b) *If k is algebraically closed, then* $U \cap$ Specmax $R^G = V \cap$ Specmax R^G.

Proof. (a) Let $P \in U$, and let $Q_1, \dots, Q_n$ be the primes lying over P. There are elements $f \in R^G \setminus P$ and $g \in R \setminus \bigcup_{i=1}^n Q_i$ such that $R_f^G \to R_g$ is étale. Let $h := \prod_{\sigma \in G} \sigma(g)$, so $h \in R^G \setminus P$. Then $R_{fh}^G \to R_h$ is étale, so $D(fh) \subseteq U$.

(b) Let $m \subseteq R^G$ be a maximal ideal.

If $m \in U$, then $R_m^G \to R_m$ is flat and module-finite, hence free of rank $\operatorname{rk}_{R_m^G} R_m = |G|$. Then $\dim_{\kappa(m)} \kappa(m) 0 \otimes_{R^G} R = |G|$ and $\kappa(m) \cong k$ is algebraically closed, so Spec $\kappa(m) \otimes_{R^G} R$ has $|G|$ elements.

If $m \in V$, then $\dim_{\kappa(m)} \kappa(m) \otimes_{R^G} R = |G|$, so R_m has $|G|$ generators over R_m^G. Also $\operatorname{rk}_{R_m^G} R_m = |G|$, so these $|G|$ generators are free generators. Thus $R_m^G \to R_m$ is flat, and its closed fiber $\kappa(m) \otimes_{R^G} R \cong \prod_{i=1}^{|G|} k$ is étale, so by [Ra, Theorem 2, p. 55], $R_m^G \to R_m$ is étale.

2. TIGHT CLOSURE

This section contains some background material on tight closure pertinent to our discussion of F-rationality in the ring R^{A_n}.

Tight closure is an operation on submodules defined principally in prime characteristic that is due to Hochster and Huneke [HH1–3, Ho1] and has been studied by many others (Fedder, Watanabe, Smith, and Aberbach, to name a few). The principal motivations for studying tight closure are results of Hochster and Huneke that settle and in some cases improve certain homological conjectures [HH1–3], and the connection made by Smith between F-rational rings (a tight closure notion) and rational singularities [S]. For example, Hochster and Huneke use tight closure, after reducing to a prime characteristic setting, to prove that direct summands of regular rings containing a field are Cohen–Macaulay [HH1]. (The proof is, actually, quite easy, given basic facts about tight closure.) They also prove a strengthened form of the Direct Summand (or "Monomial" Conjecture (again, for rings containing a field) [HH2]. Smith showed that F-rational rings have the prime-characteristic analogue of rational singularities [S]. These accomplishments provide considerable hope that a better understanding of tight closure (and of the various ring-theoretic properties associated with it) will yield further insights into homological algebra and singularities.

In its simplest form, tight closure is an operation on ideals in a ring of prime characteristic. An element x of a ring A is in the *tight closure* I^* of an ideal $I \subseteq A$ if there is an element $c \in A$ not in any minimal prime of A for which cx^q is in the ideal generated by the qth powers of elements of I, for all sufficiently high powers q of the characteristic [HH1]. Thus, taking the tight closure of an ideal produces a larger ideal.

With this definition in hand, one can now define various ring properties. We will primarily discuss F-rationality, but there are others (e.g., strong and weak F-regularity) discussed in the literature [HH1–3]. (The "F" appearing in the names of these concepts refers to the "Frobenius" map, since this map is heavily used in the tight closure operation.)

2.1. DEFINITION. A ring of prime characteristic is F-rational if all of its parameter ideals are tightly closed. A parameter ideal is one generated by elements $x_1, \dots, x_n$ that are part of a system of parameters in every localization at a prime containing them (i.e., for every prime $P \supseteq (x_1, \dots, x_n)$, the images of $x_1, \dots, x_n$ are part of a system of parameters in the localization at P) [HH3].

As mentioned earlier, K. E. Smith has shown that, under the mild hypothesis of excellence, F-rational rings have the prime-characteristic analogue of rational singularities [S]. Furthermore, F-rational rings are roughly between regular rings and Cohen–Macaulay normal rings. (That regular implies F-rational implies normal is in [HH3, HH1]. That F-rational implies Cohen–Macaulay requires a mild hypothesis (e.g., excellence or homomorphic image of Cohen–Macaulay) and is in [HH3].)

One naturally wonders about the difference between F-rationality and the property that *all* ideals are tightly closed (known as *weak F-regularity* in the literature). These two properties are generally distinct, but coincide in *Gorenstein* rings. That is, once the parameter ideals are tightly closed in a Gorenstein ring, all ideals must be tightly closed.

A large class of F-rational (or even weakly F-regular) rings is provided by the direct summands of regular rings [HH1]. This nice fact, which is evident from the definition of tight closure, also has immediate consequences for invariant rings due to the splitting property described in Section 1. Namely, we have

2.2. PROPOSITION. *Let R be a regular ring finitely generated over a field k, and let G be a finite group acting on R by k-algebra automorphisms. Suppose* char(k) *does not divide* $|G|$. *Then* R^G *is F-rational (even weakly F-regular) and Cohen–Macaulay.*

Proof. Such an R^G is a direct summand of R as an R^G-module (see Sect. 1). We have already noted that direct summands of regular rings are weakly F-regular and F-rational, due to results of [HH1]. And, we noted above, F-rational rings that are images of Cohen–Macaulay rings (for instance, finitely generated k-algebras) are themselves Cohen–Macaulay [HH3].

Thus, when a finite group G acts on a polynomial ring $R := k[X_1, \ldots, X_n]$ by permuting the variables, the invariant ring is (Cohen–Macaulay and) F-rational in characteristics relatively prime to $|G|$. So, as stated in the introduction, it is only interesting to study these properties in the finitely many characteristics that divide $|G|$.

We next define two concepts (F-purity and the a-invariant), state necessary conditions for F-rationality with these concepts, and show that a large class of invariant rings satisfy these conditions, giving evidence for their F-rationality. All rings in the remainder of this section are assumed to be of prime characteristic.

2.3. DEFINITIONS-PROPOSITION.

(a) *A ring A is* F-pure *if for any A-module M,*

$$F \otimes 1 : A \otimes M \to A \otimes M$$

is injective, where F: $A \to A$ is the Frobenius map $a \mapsto a^p$ (and p = char(A)).

(b) *An F-rational Gorenstein ring is F-pure.*

(c) *Let A be an* $\mathbf{N}$*-graded ring such that $[A]_0 = k$ is a field, and let m denote its homogeneous maximal ideal. The a-invariant of A, denoted $a(A)$, is defined to be*

$$a(A) := \max\{d \in \mathbf{Z} : [H_m^{\dim A}(A)]_d \neq 0\},$$

where the dth piece is with respect to the induced $\mathbf{Z}$*-grading on $H_m^{\dim A}(A)$.*

(d) *An F-rational graded ring (graded, in the sense of part* (c)) *has a negative a-invariant.*

Proof. See [HR2] or [HH3] for the induced $\mathbf{Z}$-grading in (c). Proofs of (b) and (d) are in [FW]. A different proof of (d) is in [HH3].

2.4. PROPOSITION. *Let R be the polynomial ring $k[X_1, \ldots, X_n]$ over the field k, and let G be a finite group that acts on R by k-algebra automorphisms.*

(a) *Suppose the action of G permutes the variables (i.e., there is an embedding of G in S_n for some n such that the action is given by $\sigma(X_i) := X_{\sigma(i)}$ for $\sigma \in G$). Then R^G is F-pure.*

(b) *Suppose the action of G preserves degrees and R^G is Cohen–Macaulay. Then $a(R^G) < 0$.*

Proof. (a) Given a monomial μ in $X_1, \ldots, X_n$, let $\phi(\mu)$ be the sum of the elements in its orbit. (Such elements will be monomials.) The reader can easily check that R^G is the k-span of the elements $\phi(\mu)$, as μ ranges over the monomials. We will construct an $(R^G)^p$-linear retraction of $(R^G)^p \subseteq R^G$ (where $p = \text{char}(k)$. This verifies the F-purity of R^G because it shows that the map $(R^G)^p \subseteq R^G$, hence also the isomorphic map F: $R^G \to R^G$, splits.

Map $R^G = k\text{-span}\{\phi(\mu)\colon \mu \text{ is a monomial}\}$ to $k\text{-span}\{\phi(\nu^p)\colon \nu \text{ is a monomial}\}$ k-linearly, by fixing the $\phi(\mu)$ for which μ is a pth power, and killing the others. Map $k\text{-span}\{\phi(\nu^p)\colon \nu \text{ is a monomial}\}$ to $k^p\text{-span}\{\phi(\nu^p)\colon \nu \text{ is a monomial}\} = (R^G)^p$ by retracting $k^p \subseteq k$ (and fixing the $\phi(\nu^p)$). The composition of these maps has the desired properties.

(b) $a(R^G)$ is independent of char(k). (The a-invariant of an **N**-graded Cohen–Macaulay ring S is the degree of the Hilbert series of S as a rational function [HH2, Discussion 7.4(b)].) R^G is F-rational in characteristics relatively prime to $|G|$ (see 2.2), so $a(R^G) < 0$ (by 2.4).

Thus, for the natural (variable permuting) action of A_n on $R = k[X_1, \ldots, X_n]$, the invariant ring R^{A_n}, which is known to be Gorenstein (see Sect. 3), is F-pure with a negative a-invariant. However, we shall see in Part II that this ring is *not* F-rational in many characteristics.

The remainder of this section mainly concerns criteria for F-rationality to be used in Part II. The following proposition contains one such criterion, as well as a result on the openness of the F-rational locus. This criterion, the initial version of which was proven by Fedder and Watanabe [FW], characterizes F-rationality in terms of F-purity with a negative a-invariant in Gorenstein rings having an isolated non-F-rational point. Each of these results can be found in greater generality in the papers cited in the proof. (For example, there is a version of (b) in [HH3] in which the Gorenstein hypothesis is weakened to Cohen–Macaulay.)

2.5. PROPOSITION. *Let A be a Gorenstein* **N**-*graded algebra finitely generated over $[A]_0 = k$, a field, and with homogeneous maximal ideal m.*

(a) *The F-rational locus is open in* Spec A, *and the radical ideal defining its complement is homogeneous.*

(b) *Suppose that k is perfect and the localization of A at any prime except m is F-rational. The following are equivalent.*

(i) *A is F-rational.*

(ii) *A is F-pure and $a(A) < 0$.*

Proof. For (a), see Theorems 4.2 and 4.4 of [HH2]. Part (b) is Theorems 7.11 and 7.12 in [HH2], except that we have replaced the F-injectivity of the Gorenstein ring A in (ii) with the equivalent condition of F-purity [HR2].

We give a second criterion, to be used in Part II, in which the F-rationality of certain Gorenstein rings is characterized by a direct summand issue.

2.6. PROPOSITION. *Let $A \subseteq B$ be a module-finite extension of rings, where A is Gorenstein and B is regular. The following are equivalent.*

(i) *A is F-rational.*

(ii) *Every ideal I of A is contracted from B, i.e., $IB \cap A = I$.*

(iii) *A is a direct summand of B as an A-module.*

Proof. (i) follows from (iii) because a direct summand of a regular ring is F-rational (even weakly F-regular) [HH1].

We now show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Assume A is F-rational. If $I \subseteq A$ is an ideal, then $IB \cap A \subseteq I^*$ because $A \subseteq B$ is module-finite [Ho1], so every ideal of A is contracted from B. (We are using that every ideal is tightly closed in an F-rational Gorenstein ring. See Section 2.) When A is Gorenstein (or even just "approximately Gorenstein," as in [Ho2]) and $A \subseteq B$ is module-finite, this contractedness condition guarantees that A is a direct summand of B [Ho2]. (That is, A is cyclically pure in B and A is approximately Gorenstein, so A is pure in B by [Ho2]. Then A must be a direct summand of B because $A \subseteq B$ is module-finite.)

Remark. Although our emphasis in this paper is on F-rationality, as opposed to weak F-regularity, we remark on the following modification to 2.6. One can replace the hypothesis of Gorenstein with normality and (i) with "A is weakly F-regular" and obtain a statement that can be proven with essentially the same proof.

3. EXAMPLES

Consider a finite group $G \leq S_n$ acting by k-algebra automorphisms on a polynomial ring $k[X_1, \ldots, X_n]$ over a field k by permuting the variables: $\sigma(X_i) := X_{\sigma(i)}$ for $\sigma \in G$. In this short section we illustrate with previously known examples that the invariant rings arising in this manner may or may not be Cohen–Macaulay or F-rational when char(k) divides the order of G. (Compare with 2.2.)

EXAMPLE 1. *The symmetric group.* Let S_n act on $k[X_1, \ldots, X_n]$ by permuting the variables. The invariant ring produced by this action is well known to be the polynomial ring $k[e_1, \ldots, e_n]$, where e_i is the ith symmetric polynomial in $X_1, \ldots, X_n$. (See [Co], for example.) This invariant ring is regular, hence Cohen–Macaulay and F-rational, regardless of the characteristic.

EXAMPLE 2. *The alternating group.* Assume $\mathrm{char}(k) \neq 2$. Let A_n act on $k[X_1, \ldots, X_n]$ by permuting the variables. The invariant ring produced here is well known to be $k[e_1, \ldots, e_n, \Delta]$, where e_i is the ith symmetric polynomial and $\Delta = \prod_{i<j}(X_i - X_j)$ is the discriminant [Co]. This ring is isomorphic to the hypersurface

$$k[e_1, \ldots, e_n, Z]/(Z^2 - \Delta^2),$$

and so is Cohen–Macaulay (even Gorenstein) in all characteristics. The F-rationality of this ring in characteristics dividing $n!/2 = |A_n|$ is studied in Part II.

EXAMPLE 3. *The cyclic group of order four in characteristic* 2. Let C_4 act on $k[X_1, \ldots, X_4]$ by cyclic permutation of the four variables. Bertin [Be] showed that the invariant ring $k[X_1, \ldots, X_4]^{C_4}$ produced here is not Cohen–Macaulay, hence also not F-rational, in characteristic 2. (Bertin, in the same paper, also shows this ring is a UFD, providing the first example of a ring known to be a UFD and not Cohen–Macaulay.)

EXAMPLE 4. *The cyclic groups of prime power p^e larger than four in characteristic p.* Fix a prime p. Let C_{p^e} act on $k[X_1, \ldots, X_{p^e}]$ by cyclic permutation of the variables. Fossum and Griffith [FG] show that the invariant ring $k[X_1, \ldots, X_{p^e}]^{C_{p^e}}$ is *not* Cohen–Macaulay in characteristic p if $p^e > 4$. (Some remarks: They also show that these rings are UFD's and that their completions at their homogeneous maximal ideals are also UFD's. Their techniques do not recover Bertin's example, in which $p^e = 4$.) Moreover, they bound the depth of $k[X_1, \ldots, X_{p^e}]$ on its homogeneous maximal ideal by

$$\operatorname{depth} k\left[X_1, \ldots, X_{p^e}\right]^{C_{p^e}} \leq p^{e-1} + 2.$$

(Note that the depth is significantly smaller than $\dim k[X_1, \ldots, X_{p^e}]^{C_{p^e}}$, which is p^e.)

In Part I, we extend their results to the general case of cyclic groups whose order is divisible by $\mathrm{char}(k)$, finishing the analysis of the Cohen–Macaulay property in $k[X_1, \ldots, X_n]^{C_n}$.

PART I: THE INVARIANT RINGS BY CYCLIC GROUPS THAT ARE NOT COHEN–MACAULAY

In Part I, we address an instance of the following question raised in the introduction to this paper.

QUESTION 1. *Let G be a finite group and fix an embedding of G as a subgroup of a permutation group S_n. This determines a k-algebra action of G on $k[X_1, \dots, X_n]$ via $\sigma(X_i) := X_{\sigma(i)}$, $\sigma \in G$. For which choices of n, $G \leq S_n$, and k is $k[X_1, \dots, X_n]^G$ Cohen–Macaulay?*

Recall from 2.2 that this question has been answered when char(k) is relatively prime to $|G|$: In this case $k[X_1, \dots, X_n]^G$ is Cohen–Macaulay. For other prior results on Question 1 see the introduction to this paper.

The instance we address is when G is the cyclic group of order n embedded in S_n in the usual way, so that a generator of C_n is identified with the n-cycle $(12 \cdots n)$. The action on $k[X_1, \dots, X_n]$ resulting from this embedding is cyclic permutation of the variables (under which $\sigma(X_1) = X_2$, $\sigma(X_2) = X_3, \dots, \sigma(X_{n-1}) = X_n$, $\sigma(X_n) = X_1$, where σ is a generator of C_n).

We show in Theorem 7.2 that the invariant ring $k[X_1, \dots, X_n]^{C_n}$ resulting from this action is *not* Cohen–Macaulay in the characteristics dividing n for $n > 4$. Because the Cohen–Macaulay-ness of this ring is already known for the other values of char(k) and n (see 2.2 and Sect. 3), Theorem 7.2 completes the analysis of the Cohen–Macaulay property in $k[X_1, \dots, X_n]^{C_n}$. In other words, we have answered Question 1 for the natural embedding of C_n in S_n for any n and any k.

We achieve 7.2 by proving a result (Theorem 4.1) concerning the nonfree locus (see Definition 1.2) of the action of *any* finite abelian group G on very general rings R in characteristics dividing $|G|$. This result asserts that, under suitable hypotheses, the defining ideal J of this locus must have small *depth* in R or in R^G.

From this theorem come two natural corollaries. First, assuming it is the case that J has small depth in R^G, one derives in the usual way a corresponding bound on the depth of R^G in terms of the height of the ideal J in the graded case (Corollary 5.2). Second, because heights are preserved in expanding from R^G to R, J must have small height *if R and R^G are both Cohen–Macaulay* (Corollary 5.1).

Roughly speaking, each of these corollaries says that assuming R is Cohen–Macaulay, R^G will not be Cohen–Macaulay if J has large height. Thus, the goal of understanding the Cohen–Macaulay property in R^G leads us to the problem of computing the height of J. This computation is easily carried out in the situation of 7.2, giving our result on the characteristics in which cyclic permutation of the variables in a polynomial ring produces a Cohen–Macaulay invariant ring. If one could show that J has "large height" for other embeddings $G \leq S_n$ where G is abelian, one could similarly answer Question 1 in these instances: The resulting invariant rings $k[X_1, \dots, X_n]^G$ would not be Cohen–Macaulay in characteristics dividing $|G|$.

Section 7 also contains generalizations of 7.2 in which we find analogous results for more general rings R (Theorem 7.1) or more general groups G (Theorem 7.3).

4. THE DEPTH OF THE NONFREE LOCUS

This theorem finds that, for quite general actions of finite abelian groups G on affine algebras R, the defining ideal of the locus where the action is not free (see Definition 1.2) has small depth in either the invariant ring R^G, or in R itself (i.e., the expansion of the ideal to R has small depth).

Before giving the proof, we would like to relate Theorem 4.1 to the main result, Proposition 4, of [Fo], which also bounds the depth of this same ideal (the defining ideal of the nonfree locus) for certain actions by groups on rings. Basically, the comparison is this: Their proofs use similar techniques, but they study different types of group actions.

In [Fo] it is assumed that G is cyclic, R is local, and that action of G is free on the punctured spectrum of R (i.e., G acts freely on all fibers, except, possibly, on the one consisting of the maximal ideal of R). In contrast, for our result, G is any finite abelian group, R is an affine algebra, and we make no assumptions about where G acts freely. (Additional hypotheses in the two theorems coincide.)

The lack of assumptions on the size of the nonfree locus is a critical distinction, for in many of the situations in which 4.1 could be applied, this locus is rather large. As an example, consider the case in which $G = C_n$ permutes the n variables in a polynomial ring $R = k[X_1, \dots, X_n]$, k is algebraically closed, and $n > 1$. $R^G \subseteq R$ has many one-element fibers: The fiber containing a maximal ideal of the form $(X_1 - \lambda, \dots, X_n - \lambda)$, where λ is in k, can contain no other ideal because $(X_1 - \lambda, \dots, X_n - \lambda)$ is fixed under the action. (See Section 1.) And the contractions to R^G of *each* of these ideals are in the nonfree locus of Spec R^G. (Their fibers are too small for them to be in the free locus: See 1.4.) Thus, the nonfree locus has considerable size in this example.

However, the basic technique used in the proof of Theorem 4.1 is very similar to that used in Proposition 4 of [Fo]. Namely, one chooses the ideal I in the spectral sequences of Proposition 1.1 to make many terms vanish, thereby obtaining information about depths in R^G. Because of his hypotheses, Fogarty is able to obtain such information on taking I to be the maximal ideal of his local ring R^G. We choose the ideal I for our situation from the following two observations. If I is generated by elements that form a regular sequence in R, then many terms of **I** will vanish because of the sheaf cohomology on $\dot{X}$. If I is contained in the defining ideal of the nonfree locus, many terms of **II** will, as it turns out, vanish because of our normal basis theorem in Proposition 1.3.

As a final note concerning Proposition 4 of [Fo], we observe that, even in the case in which G is cyclic, one cannot simply localize the rings R to which our theorem applies to obtain a situation in which Fogarty's theorem could be applied. To illustrate our point, consider the special case of cyclic variable permutation in a polynomial ring. This situation has the advantage that there is a minimal prime $P \subseteq R^G$ of the defining ideal of the nonfree locus (in $\operatorname{Spec} R^G$) having a single prime Q in its fiber in $\operatorname{Spec} R$, as one sees from 6.2. Thus, G acts freely on the punctured spectrum of the local ring R_Q, so one is tempted to apply [Fo] to R_Q, in the hope of showing R_P^G is not Cohen–Macaulay. However, a crucial hypothesis of Fogarty's Proposition 4 is missing: that G act trivially on R_Q/QR_Q. (Using 6.2, this ring is seen to be of the form $k(X_1, \dots, X_{n/q_j})$, where q_j divides n, a ring on which the variable permuting action is nontrivial.)

4.1. THEOREM. *Let R be a domain finitely generated over a field k, and suppose a finite abelian groiup G acts on R by by k-algebra automorphisms. Assume*

(1) $\operatorname{char}(k)$ *divides* $|G|$, *and*

(2) *there is a G-stable maximal ideal $\tilde{m}$ for which G acts trivially on $R/\tilde{m}$.*

Let $J \subseteq R^G$ be an ideal defining the locus in Specmax R^G *where the action of G is not free (see* 1.2). *Then* $\operatorname{depth}_J R^G \leq 2$ *or* $\operatorname{depth}_{JR} R \leq 2$.

Proof. Our first objective is to show that the group cohomology $\mathrm{H}^1(G, R)$ is nontrivial, as a consequence of (1) and (2). To do this, we first note that k, considered as a $k[G]$-module via the trivial G-action, splits off from R in the category of $k[G]$-modules. For we have $k[G]$-linear maps

$$k \to R \to R/\tilde{m} \to k,$$

where $k \to R$ is the natural inclusion, $R \to R/\tilde{m}$ is the natural surjection, and $R/\tilde{m} \to k$ is a k-linear splitting of k from its extension field $R/\tilde{m}$. (We are using that G acts trivially on $R/\tilde{m}$ when we claim that the map $R/\tilde{m} \to k$ is $k[G]$-linear, since G acts trivially on k.) Also, the composition $k \to k$ is the identity. Consequently, $\mathrm{H}^1(G, k)$ is a direct summand (as a $k[G]$-module) of $\mathrm{H}^1(G, R)$, so it is enough to see that $\mathrm{H}^1(G, k)$ is nonzero.

Let N denote the kernel of the $k[G]$-linear map $k[G] \to k$ sending 1 to 1 (and so σ is sent to 1 for all $\sigma \in G$, because k has the trivial G-action). Then N is a first module of syzygies of k as a $k[G]$-module, so $\mathrm{H}^1(G, k)$

$= \mathrm{Ext}^1_{k[G]}(k,k) = \mathrm{Hom}_{k[G]}(N,k)$. Thus, to see $\mathrm{H}^1(G,k) \neq 0$, it suffices to find a nonzero $k[G]$-linear map $N \to k$.

We now construct such a map. Let $N_{\mathbf{Z}}$ denote the kernel of the $\mathbf{Z}[G]$-linear map $\mathbf{Z}[G] \to \mathbf{Z}$ under which 1 maps to 1 (where $\mathbf{Z}$ is given the trivial G-action). $N_{\mathbf{Z}}$ is a free $\mathbf{Z}$-module with basis given by the elements $\sigma - 1$, as σ ranges over the elements of the group G [HS].

Fix a subgroup H of G of index p. Consider the composition of the $\mathbf{Z}$-linear surjection $N_{\mathbf{Z}} \to G$, given by $\sigma - 1 \mapsto \sigma$, with the quotient map $G \to G/H \cong \mathbf{Z}/p\mathbf{Z}$. The composition is a nonzero $\mathbf{Z}$-linear map

$$N_{\mathbf{Z}} \to \mathbf{Z}/p\mathbf{Z}$$

sending $\sigma - 1$ to an element $n(\sigma + H) \in \mathbf{Z}/p\mathbf{Z}$ corresponding to the coset $\sigma + H$.

Tensoring with k now produces a k-linear map

$$\phi : k \otimes_{\mathbf{Z}} N_{\mathbf{Z}} \to k \otimes_{\mathbf{Z}} \mathbf{Z}/p\mathbf{Z} \cong k,$$

and $k \otimes_{\mathbf{Z}} N_{\mathbf{Z}}$ is isomorphic to N as $k[G]$-module. (The obvious k-vector space isomorphism between the two is $k[G]$-linear as well.) Thus, we have a map $\phi : N \to k$. This map is easily checked to be $k[G]$-linear and to assign a nonzero value to any element $\tau - 1$ whenever $\tau \in G \setminus H$. This completes the argument that $\mathrm{H}^1(G,R) \neq 0$.

Now, suppose $\mathrm{depth}_J R^G \geq 3$ and $\mathrm{depth}_{JR} R \geq 3$. Using prime avoidance, find elements u_1, u_2, u_3 in J that are a regular sequence both in R^G and in R. Let I be the ideal (u_1, u_2, u_3).

Let $X = \mathrm{Spec}\, R$, $Y = \mathrm{Spec}\, R^G$, $\dot{X} = X \setminus V(IR)$, and $\dot{Y} = Y \setminus V(I)$, and let $\mathbf{I}, \mathbf{II}$ denote the spectral sequences from Proposition 1.1,

$$\mathbf{I} : H^q\big(G, H^p\big(\dot{X}, \mathscr{O}_X\big)\big)$$

$$\mathbf{II} : H^p\big(\dot{Y}, \mathbf{H}^q(G, \mathscr{O}_X)\big),$$

so $\mathbf{H}^q(G, \mathscr{O}_X)$ is the sheaf associated to the presheaf on $\dot{Y}$ that sends an open set U to $H^q(G, H^0(V, \mathscr{O}_X))$, where V is the inverse image of U under $\dot{X} \to \dot{Y}$.

Our next objective is to use the normal basis theorem (1.3) to show that the higher sheaves $\mathbf{H}^q(G, \mathscr{O}_X)$ necessarily vanish. Fix $q > 0$ and let $\mathscr{F}$ denote the presheaf on $\dot{Y}$ described in the last paragraph (from which we obtained $\mathbf{H}^q(G, \mathscr{O}_X)$). Fix a maximal ideal m in $\dot{Y}$, and let $\{n_i\}_i$ be the (finitely many) maximal ideals lying over m. We will show that the stalk of $\mathscr{F}$ at m is zero. (Then $\mathbf{H}^q(G, \mathscr{O}_X) = 0$ because it has the same stalks.)

$R^G_m \to R_m$ is étale, so we may apply Proposition 1.3 to the semilocal domain $R_m = (R^G \setminus m)^{-1} R \cong (R \setminus \bigcup n_i)^{-1} R$. This proposition gives an *a* and b in R (with $b \notin \bigcup_i n_i$) such that

$$(R \setminus \bigcup_i n_i)^{-1} R = \oplus_{i=1}^{|G|} R_m^G \frac{\sigma^i(a)}{\sigma^i(b)}.$$

Because R is module-finite over R^G, we can then find a $g \in R^G \setminus m$ such that

$$R_g = \oplus_{i=1}^{|G|} R_g^G \frac{\sigma^i(a)}{\sigma^i(b)}.$$

i.e., $R_g \cong R_g^G \otimes_k k[G]$ as $k[G]$-modules.

We now proceed to show that $\mathscr{F}_m = \varinjlim_{f \notin m} H^q(G, R_f)$ is zero. Given $f \in R^G \setminus m$, we are to find an $h \in (fR^G) \setminus m$ such that $H^q(G, R_h) = 0$. Let $h := fg$, with g as above. Then

$$R_h \cong R_h^G \otimes_k k[G]$$

as a $k[G]$-module, and so is free over $k[G]$. R_h is then also injective over $k[G]$ because $k[G]$ is an Artin Gorenstein ring. (The group ring $k[G]$ of a finite abelian group G over a field k is well known to be Artin Gorenstein. One way to see this is the following: The group ring of a cyclic group is a hypersurface, and the group ring of a product of groups is the tensor product of the group rings. In any case, it is also in [CR].) Consequently,

$$H^q(G, R_h) = \operatorname{Ext}^q_{k[G]}(k, R_h^G \otimes_k k[G]) = 0.$$

This completes the argument that the higher $\mathbf{H}^q(G, \mathscr{O}_X)$ vanish. Hence, we now know that the terms $\mathbf{II}^{pq}$ of the spectral sequence $\mathbf{II}$ also vanish for $q > 0$.

We next note that $\mathbf{H}^0(G, \mathscr{O}_X) \cong \mathscr{O}_Y$, as sheaves on $\dot{Y}$. Let $\mathscr{F}$ be the presheaf on $\dot{Y}$ from which we obtained $\mathbf{H}^0(G, \mathscr{O}_X)$. For each basis element $D(f) \in \dot{Y}$, we have isomorphisms

$$\mathscr{F}(D(f)) \cong R_f^G \cong \mathscr{O}_Y(D(f))$$

that commute with the restriction maps on $\mathscr{F}$ and $\mathscr{O}_Y$, so they induce an isomorphism of the sheaves

$$\mathbf{H}^0(G, \mathscr{O}_X) \cong \mathscr{O}_Y.$$

We have now shown that the spectral sequence $\mathbf{II}$ collapses, in the sense that $\mathbf{II}^{pq} = 0$ for $q > 0$. Thus, the E_2^{pq} terms $\mathbf{II}^{pq}$ are also the E_∞^{pq} terms for the sequence $\mathbf{II}$, since the sequence $\mathbf{II}$ comes from a double complex.

Since the spectral sequence **I** has the same abutment as **II**, and the latter converges to $\mathbf{II}^{n0} \cong H^n(\dot{Y}, \mathcal{O}_Y)$, we have $\mathbf{I}^{pq} \overset{q}{\Rightarrow} H^n(\dot{Y}, \mathcal{O}_Y)$.

Because the ideal I is generated by a elements that form a regular sequence in R, the nonzero terms among the $\mathbf{I}^{pq}$ occur in only two rows. More precisely,

$$\mathbf{I}^{pq} = \begin{cases} 0 & \text{if } p \neq 0,2 \\ H^q(G, H_I^3(R)) & \text{if } p = 2 \\ H^q(G, R) & \text{if } p = 0. \end{cases}$$

We next use the above information to show that the term E_∞^1 of the sequence **I** is isomorphic to the first group cohomology $H^1(G, R)$. Since q is the filtration degree for **I**, the boundary maps are

$$d_r^{pq} : E_r^{pq} \to E_r^{p-r+1, q+r},$$

and $d_r^{pq} = 0$ unless $p = 2$ and $r = 3$. Thus, the E_4 term of this spectral sequence is the E_∞ term, and in each spot of this complex there are at most two nonzero graded pieces:

$$E_\infty^n \cong E_4^n = E_4^{2,n-2} \oplus E_4^{0n}.$$

E_∞^1 has, at most, a single nonzero graded piece (because $E_4^{2,-1} = 0$):

$$E_\infty^1 = E_\infty^{01} \cong E_4^{01}.$$

But E_4^{01} is the cohomology at E_3^{01}

$$\ldots 0 = E_3^{2,-2} \xrightarrow{d_3^{2,-2}} E_3^{01} \xrightarrow{d_3^{01}} \ldots,$$

and both of these maps are zero, so $E_4^{01} \cong E_3^{01}$, which, in turn, is isomorphic to $E_2^{01} = \mathbf{I}^{01}$ because $d_2 = 0$. Thus,

$$E_\infty^1 \cong \mathbf{I}^{01} = H^1(G, R).$$

Writing F_i for the ith filtered piece of the filtered module $E_\infty^1 \cong \mathbf{I}^1$, we have shown

$$H^1(G, R) \cong E_\infty^{01} = \frac{F_1}{F_2}$$

(since the index q indicates the filtration degree). Recalling $\mathbf{I}^{pq} \overset{q}{\Rightarrow} H^n(\dot{Y}, \mathcal{O}_Y)$, we see that F_1/F_2 is a graded piece of an associated graded of the module $H^1(\dot{Y}, \mathcal{O}_Y)$. Since the filtration of a double complex is finite in each spot (i.e., the F_i are eventually zero) we must have that $F_2 = 0$ and $F_1 \cong H^1(\dot{Y}, \mathcal{O}_Y)$. Therefore,

$$H^1(\dot{Y}, \mathcal{O}_Y) \cong F_1 \cong H^1(G, R).$$

But $H^1(\dot{Y}, \mathcal{O}_Y) \cong H_I^2(R^G) = 0$ because I has depth three, and we showed that $H^1(G, R) \neq 0$. Thus, it must have been the case that $\operatorname{depth}_J R^G \leq 2$ or that $\operatorname{depth}_{JR} R \leq 2$.

5. COROLLARIES TO THE MAIN THEOREM OF PART I

In this section, we derive two corollaries from Theorem 4.1 that will be applied in Section 7, where we discuss the Cohen–Macaulay property in the invariant ring by cyclic variable permutation. The first of these (5.1) says that a certain ideal (J) must have small height if R and R^{C_n} are Cohen–Macaulay.

5.1. COROLLARY. *Let R, G, and J be as in Theorem* 4.1. *If R and R^G are Cohen–Macaulay, then* $\operatorname{ht} J \leq 2$.

Proof. $\operatorname{ht} J = \operatorname{ht} JR$ (because $R^G \subseteq R$ is module-finite), and this is equal to $\operatorname{depth}_J R^G$ and $\operatorname{depth}_{JR} R$ if R and R^G are Cohen–Macaulay.

The second corollary (5.2) obtains a bound for the depth of the invariant ring in the graded case.

5.2. COROLLARY. *Let R, G, k, and J be as in Theorem 4.1. Suppose R is an* $\mathbf{N}$*-graded k-algebra with $[R]_0 = k$, and the action of G on R preserves degrees (so R^G is also* $\mathbf{N}$*-graded). Assume* $\operatorname{depth}_{JR} R > 2$. *Then*

$$\operatorname{depth} R^G \leq \dim R - \operatorname{ht} J + 2,$$

where $\operatorname{depth} R^G$ *is taken on the homogeneous maximal ideal of R^G.*

Proof. We can add $\dim R^G - \operatorname{ht} J$ elements to J to generate the homogeneous maximal ideal of R^G up to radicals. Thus, $\operatorname{depth} R^G \leq \operatorname{depth}_J R^G + (\dim R^G - \operatorname{ht} J)$. (The depth increases by at most one for each element added to the supporting ideal, due to Nakayama's Lemma.) The result now follows from $\dim R^G = \dim R$ (R is module-finite over R^G) and $\operatorname{depth}_J R^G \leq 2$ (which we know from 4.1).

6. A BOUND ON THE CODIMENSION OF THE NONFREE LOCUS FOR CYCLIC VARIABLE PERMUTATION

The two corollaries in the previous section lead us from the problem of studying the Cohen–Macaulay property in rings R^G satisfying the hypotheses of 4.1 to the problem of computing the codimension of their nonfree locus (i.e., computing the height of the ideal J in 4.1). In this section, we show a lower bound for this codimension in a special case (Proposition 6.2). This is the case in which $G = C_n$ cyclically permutes the images of $X_1, \dots, X_n$ in a graded domain $k[X_1, \dots, X_n]/P$ and k is algebraically closed. (P is, of course, chosen such that this action is well-defined.) The other proposition in this section (6.1) reduces the Cohen–Macaulay issue in invariant rings of k-algebras to the case in which k is algebraically closed. In Section 7 we will combine these results with the corollaries from Section 5 to analyze the Cohen–Macaulay property in invariant rings like those described above, but where k is not required to be algebraically closed.

6.1. PROPOSITION. *Let R be a ring that is $\mathbf{N}$-graded and finitely generated over $[R]_0 = k$, a field. and let $\overline{R} := \overline{k} \otimes_k R$, where $\overline{k}$ is an algebraic closure of k. Let G be a finite group that acts on R by degree-preserving k-algebra automorphisms. Then*

$$\operatorname{depth} R^G = \operatorname{depth} \overline{R}^G,$$

where $\overline{R}^G$ refers to the induced $\overline{k}$-algebra action of G on $\overline{R}$ and each depth is on the homogeneous maximal ideal.

Proof. The induced $\overline{k}$-algebra action of G on $\overline{R}$ is that determined by $\sigma(\alpha \otimes r) := \alpha \otimes \sigma(r)$ for $\sigma \in G$, $\alpha \in \overline{k}$, $r \in R$.

We first note that $\overline{k} \otimes_k R^G \cong \overline{R}^G$ as $\overline{k}$-algebras. R^G is the intersection of the k-modules

$$R^G = \textstyle\bigcap_{\sigma \in G} \ker(\sigma - 1_G : R \to R)$$

(where in writing $\sigma - 1_G : R \to R$ we are identifying the elements of G with the k-algebra automorphisms they induce on R). Commuting the faithfully flat base change $\overline{k} \otimes_k$ with the above finite intersection gives

$$\begin{aligned} \overline{k} \otimes R^G &= \textstyle\bigcap_{\sigma \in G} \overline{k} \otimes \ker(\sigma - 1_G : R \to R) \\ &= \textstyle\bigcap_{\sigma \in G} \ker\left(\sigma - 1_G : \overline{R} \to \overline{R}\right) \\ &= \overline{R}^G. \end{aligned}$$

The composition is a $\overline{k}$-algebra isomorphism $\overline{k} \otimes R^G \cong \overline{R}^G$.

Thus, $\bar{R}^G$ is flat over R^G, and the homogeneous maximal ideal of $\bar{R}^G$ is the expansion of that of R^G. Therefore,

$$\operatorname{depth} \bar{R}^G = \operatorname{depth} R^G.$$

(Explanation: If m is the homogeneous maximal ideal of R^G and $x_1, \ldots, x_n$ is a maximal R^G-sequence in m, then the images of $x_1, \ldots, x_n$ are an $\bar{R}^G$-sequence in $m\bar{R}^G$, and R^G/m injects into $R^G/(x_1, \ldots, x_n)$, R^G-linearly. But then $\bar{R}^G/m\bar{R}^G$ injects into $\bar{R}^G/(x_1, \ldots, x_n)\bar{R}^G$, $\bar{R}^G$-linearly, so $x_1, \ldots, x_n$ is a maximal $\bar{R}^G$-sequence in $m\bar{R}^G$, the homogeneous maximal ideal of $\bar{R}^G$.)

We now compute the bound on the nonfree locus for cyclic variable permutation over an algebraically closed field.

6.2. PROPOSITION. *Let G be the cyclic group of order n with generator σ. Let R be the ring*

$$R := \frac{k[X_1, \ldots, X_n]}{P},$$

where k is an algebraically closed field and P is a homogeneous prime that is stable under the k-algebra action of G under which $\sigma(X_i) := X_{\sigma(i)}$.

Consider the induced k-algebra action of G on R that permutes the images of $X_1, \ldots, X_n$. Let V denote the nonfree locus in Specmax R^G *(see 1.2 for the definition of this locus) and let Z denote its inverse image in* Spec R, *which lies in* Specmax R *since $R^G \subseteq R$ is module-finite. Then Z is defined by*

$$\bigcap_{j=1}^{v} \left(\pi(X_i) - \pi(X_{n/q_j + i}) : 1 \le i \le (q_j - 1)\frac{n}{q_j} \right),$$

where $q_1, \ldots, q_v$ are the primes dividing n, and $\pi(X_i)$ denotes the image of X_i in R. Furthermore, we have

$$\operatorname{ht} J \ge \dim R - \frac{n}{p_0},$$

where $p_0 = \min\{q_1, \ldots, q_v\}$, and $J \subseteq R^G$ is a defining ideal for V.

Proof. We first find the defining ideal of Z, using the description of the nonfree locus from 1.4. Let $\pi : k[X_1, \ldots, X_n] \to R$ be the natural surjection. Fix a maximal ideal $m_\lambda := (X_1 - \lambda_1, \ldots, X_n - \lambda_n)$ containing P, where $\lambda_1, \ldots, \lambda_n \in k$. We are to show that $\operatorname{Stab}_G \pi(m_\lambda)$ is nontrivial if and only if m_λ contains the prescribed ideal.

The cyclic group $G = C_n$ has as its minimal nontrivial subgroups

$$\langle \sigma^{n/q_1} \rangle, \ldots, \langle \sigma^{n/q_r} \rangle,$$

so, $\mathrm{Stab}_G \pi(m_\lambda)$ is nontrivial precisely when it contains one of these cyclic subgroups, i.e., when there exists j such that $\sigma^{n/q_j}(m_\lambda) \equiv m_\lambda \bmod P$. But P is contained in both of these ideals m_λ and $\sigma^{n/q_j}(m_\lambda)$ (using that P is G-stable). Thus, $\mathrm{Stab}_G \pi(m_\lambda) \neq 1$ if and only if there is a j for which

$$\sigma^{n/q_j} m_\lambda = m_\lambda,$$

i.e., every n/q_jth coordinate of the vector $(\lambda_1, \ldots, \lambda_n)$ is the same, i.e.,

$$m_\lambda \supseteq \left(X_i - X_{n/q_j + i} : 1 \leq i \leq (q_j - 1)\frac{n}{q_j} \right),$$

i.e., $\pi(m_\lambda) \supseteq (\pi(X_i) - \pi(X_{n/q_j+i}) : 1 \leq i \leq (q_j - 1)(n/q_j))$. Thus, the defining ideal is as claimed.

We next show the lower bound on the height of J. First note $\mathrm{ht}\, J = \mathrm{ht}(JR)$ because $R^G \subseteq R$ is module-finite. Now, let Q_j denote the ideal

$$\left(\pi(X_i) - \pi\left(X_{n/q_j + i}\right) : 1 \leq i \leq (q_j - 1)\frac{n}{q_j} \right).$$

We have $\mathrm{ht}(JR) = \mathrm{ht}(\bigcap_j Q_j)$ because these two ideals define the same closed set in Specmax R and R is finitely generated over a field (so that these two ideals must have the same radical, and hence the same height). Thus, it suffices to show $\mathrm{ht}(\bigcap_j Q_j) \geq \dim R - n/p_0$, i.e., that $\mathrm{ht}\, Q_j \geq \dim R - n/p_0$ for all j. But, Q_j produces a quotient ring R/Q_j that is itself a quotient of $k[X_1, \ldots, X_n]/(X_i - X_{i+n/q_j} : 1 \leq i \leq (q_j - 1)n/q_j)$ $\cong k[X_1, \ldots, X_{n/q_j}]$. Thus, $\mathrm{ht}\, Q_j \geq \dim R - n/q_j$, as desired.

7. THE CHARACTERISTICS IN WHICH CYCLIC VARIABLE PERMUTATION PRODUCES A COHEN–MACAULAY INVARIANT RING

This section contains one of the two main results of the paper—a complete description of the characteristics in which cyclic permutation of the variables in a polynomial ring produces a Cohen–Macaulay invariant ring (7.2).

Indeed, we show a more general result (7.1) that accomplishes the same task (i.e., describes the characteristics in which the Cohen–Macaulay property holds) for a larger class of invariant rings, except in low dimensions. As it happens, the Cohen–Macaulayness of the "low dimensional" examples arising in the situation of 7.2 is known from Section 3, so that 7.2 will follow directly from 7.1.

We also prove a generalization of 7.2 to products of cyclic groups acting on a polynomial ring by permuting disjoint variables (7.3).

We first make an easy observation, to be used in Theorem 7.1, that strengthens Proposition 2.2. This observation is basically that in the graded case, one can weaken the hypothesis that R be regular to R just being Cohen–Macaulay and still conclude that R^G is Cohen–Macaulay. That is, when a finite group G acts by ring automorphisms on a Cohen–Macaulay graded ring R and $|G|$ is invertible in R, then R^G is Cohen–Macaulay. To see this, note that a homogeneous system of parameters for R^G remains a homogeneous system of parameters in R (because $R^G \subseteq R$ is module-finite). This system is then a regular sequence in R, and hence in R^G, because ideals of the direct summand R^G are contracted from R.

Thus, under very general hypotheses, if R is Cohen–Macaulay and char(R) does not divide $|G|$, then R^G is Cohen–Macaulay. Theorem 7.1 presents a sort of converse: It states that, for certain actions on Cohen–Macaulay rings R, R^G can *only* be Cohen–Macaulay in these special characteristics. In addition, 7.1 bounds the depth when R^G is not Cohen–Macaulay. This bound is usually considerably smaller than the dimension, a point that is illustrated in the application 7.2.

7.1. THEOREM. *Let G be a cyclic group of order n, with generator σ. Let R be a ring of the form*

$$R := \frac{k[X_1, \dots, X_n]}{P},$$

where k is a field and P is a homogeneous prime stable under the k-algebra action of G under which $\sigma(X_i) := X_{\sigma(i)}$.

Consider the induced k-algebra action of G on R that permutes the images of $X_1, \dots, X_n$. Assume

(1) *R is Cohen–Macaulay,*

(2) *$\bar{k} \otimes_k R$ is a domain, where $\bar{k}$ is an algebraic closure of k (i.e., $P\bar{k}[X_1, \dots, X_n]$ is prime), and*

(3) dim $R > 2 + n/p_0$, *where p_0 is the smallest prime dividing n.*

Then R^G is Cohen–Macaulay if and only if char(k) *is relatively prime to n.*

Furthermore, if R^G is not Cohen–Macaulay, then depth $R^G \le 2 + n/p_0$.

Remark. Note that even if the dimension of R^G (i.e., dim R) is significantly larger than $2 + n/p_0$, its depth is still $\le 2 + n/p_0$ when R^G is not Cohen–Macaulay.

Proof. The "if" part follows from the paragraphs preceding this theorem.

Now, assume char(k) divides n. Using 6.1, reduce to the case in which k is algebraically closed. Let J be as in Theorem 4.1. (That is, J defines the nonfree locus in Specmax R^G.) In 6.2, we computed ht $J \geq \dim R - n/p_0$, so the result now follows from Corollaries 5.2 and 5.1. (In these corollaries, which require the hypotheses of Theorem 4.1, we may take $\tilde{m}$ to be the homogeneous maximal ideal of R.)

We isolate the case $P = (0)$ of Theorem 7.1, yielding one of the major results of the paper: an analysis of the Cohen–Macaulay property in invariant rings of polynomial rings by cyclic permutation of the variables. Again, this extends results of [FG], which contains the special case of 7.2 in which n is a power of char(k).

7.2. THEOREM. *Consider the k-algebra action of the cyclic group C_n of order n on the polynomial ring $k[X_1, \ldots, X_n]$ that cyclically permutes the variables. (That is, consider the k-algebra action determined by*

$$\begin{aligned} \sigma(X_1) &= X_2 \\ \sigma(X_2) &= X_3 \\ &\vdots \\ \sigma(X_{n-1}) &= X_n \\ \sigma(X_n) &= X_1, \end{aligned}$$

where σ generates C_n.)

Then $k[X_1, \ldots, X_n]^{C_n}$ is Cohen–Macaulay if and only if char(k) *is relatively prime to n or $n \leq 3$.*

Furthermore, if $k[X_1, \ldots, X_n]^{C_n}$ is not Cohen–Macaulay, then

$$\operatorname{depth} k[X_1, \ldots, X_n]^{C_n} \leq 2 + \frac{n}{p_0},$$

where p_0 is the smallest prime dividing n.

Remark. Note that the bound on the depth of $k[X_1, \ldots, X_n]^{C_n}$ when this ring is not Cohen–Macaulay is usually significantly smaller than the dimension, which is n.

Proof. Recall from 2.2 (or from the paragraphs preceding 7.1) that $k[X_1, \ldots, X_n]^{C_n}$ is Cohen–Macaulay when char(k) does not divide n.

As for the examples where $n \leq 4$ and char(k) divides n, see Section 3.

The rest (i.e., $n > 4$ and char(k) divides n) follows from 7.1. That is, if $n > 4$, then $n = \dim k[X_1, \ldots, X_n]$ is easily checked to be larger than

$2 + n/p_0$. Theorem 7.1 then applies to give that $k[X_1, \ldots, X_n]^{C_n}$ is not Cohen–Macaulay, and depth $k[X_1, \ldots, X_n]^{C_n} \leq 2 + n/p_0$. (Note that this bound is strictly smaller than the dimension when $n > 4$.)

Another natural (and virtually immediate) generalization of Theorem 7.2, is to certain actions by *products* of cyclic groups.

7.3. THEOREM. *Consider the action of a finite abelian group*

$$G = C_1 \times \ldots \times C_t$$

that "permutes disjoint variables" in the polynomial ring $k[X_1, \ldots, X_n]$ *where* $n = \sum_i |C_i|$. *That is, embed* G *in* S_n *so that the generator of* C_i *is identified with the permutation*

$$\left(\sum_{j=1}^{i-1} |C_j| + 1 \sum_{j=1}^{i-1} |C_j| + 2 \ldots \sum_{j=1}^{i} |C_j| \right),$$

and then let G *act on* $k[X_1, \ldots, X_n]$ *according to the* k*-algebra action determined by* $(\sigma_1 \times \ldots \times \sigma_t)(X_i) = X_{\sigma_1 \ldots \sigma_t(i)}$, *where* $\sigma_j \in C_j$ *for all* j.
Then $k[X_1, \ldots, X_n]^G$ *is Cohen–Macaulay if and only if*

(1) $\operatorname{char}(k)$ *is relatively prime to* $|G|$, *or*

(2) $\operatorname{char}(k) = 2$ *or* 3, *and every factor* C_j *whose order is divisible by* $\operatorname{char}(k)$ *has order equal to* $\operatorname{char}(k)$.

Proof. We have

$$k[X_1, \ldots, X_n]^G \cong R_1^{C_1} \otimes_k \ldots \otimes_k R_t^{C_t},$$

where $R_i \subseteq k[X_1, \ldots, X_n]$ is the polynomial ring in the $|C_i|$ variables on which C_i acts. Specifically, $R_i = k[X_{\sum_{j=1}^{i-1}|C_j|+1}, \ldots, X_{\sum_{j=1}^{i}|C_j|}]$. Thus, $k[X_1, \ldots, X_n]^G$ is Cohen–Macaulay if and only if each $R_i^{C_i}$ is. The result now follows from Theorem 7.2.

Remark. One could also try to obtain 7.3 by computing the codimension of the nonfree locus of the action. (In other words, one could try to give a proof of 7.3 that is analogous to the proof of 7.2.) However, this only gives the partial result that if $k[X_1, \ldots, X_n]^G$ is Cohen–Macaulay and $G \neq 1$, then G must have a factor of C_2 or C_3. One cannot tell from this information whether, for instance, the group $G = C_3 \times C_9$ produces a Cohen–Macaulay invariant ring in characteristic 3 (whereas we know from 7.3 that this invariant ring is *not* Cohen–Macaulay).

Note that Theorem 7.3 does *not* answer Question 1 for finite abelian groups G. That is, Theorem 7.3 only answers Question 1 for a *particular* *type* of embedding of a finite abelian group G into a permutation group, namely, one in which the generators of the cyclic factors are identified with *cycles involving disjoint letters*.

Of course, there can be more than one embedding of this type for a given group G. For instance, $C_6 \cong C_2 \times C_3$ may be embedded into S_6 by identifying its generator with the cycle (123456), or into S_5 by identifying a generator of C_2 with (12) and a generator of C_3 with (345).

There are also embeddings that are not of this type. As an example, consider the group

$$G = C_2 \times C_2$$

and embed it into S_4 by identifying an element $\sigma \in G$ with the permutation of the four elements of G given by multiplication by σ. Explicitly, this identifies the generator of one factor of C_2 with (12)(34) and the generator of the other factor of C_2 with (13)(24), up to a reordering of the letters. This embedding of G into S_4 determines an action of G on $k[X_1, X_2, X_3, X_4]$ as described in Question 1. We now ask: Is $k[X_1, X_2, X_3, X_4]^G$ Cohen–Macaulay? Theorem 7.3 is of no use here because it does not apply to this type of embedding of G into S_4.

Thus, given a finite abelian group G, we can answer Question 1 for *certain* types of embedding of G in a permutation group. What about for other embeddings? Theorem 4.1 might be of use: What is needed is a computation of the codimension of the nonfree locus (i.e., the height of J as in 4.1). If this height is large enough, then the corollaries of Section 5 will give information about the Cohen–Macaulayness of $k[X_1, \dots, X_n]^G$.

Although we have focused on the application of 7.1 to polynomial rings (i.e., the special case of 7.1 in which $P = (0)$), it is also interesting to apply this theorem to $P \neq (0)$ of small height. For a simple example, fix a homogeneous polynomial $h \in k[X_1, \dots, X_n]$ that is irreducible over the algebraic closure of k and invariant under cyclic permutation of the variables. (For instance, $h = X_1^2 + \cdots + X_n^2$ will do when $\text{char}(k) \neq 2$ and $n \geq 3$.) Now, consider the action of C_n on

$$R := \frac{k[X_1, \dots, X_n]}{(h)}$$

that permutes the images of $X_1, \dots, X_n$. Theorem 7.1 tells us that for $n > 6$, the invariant ring R^{C_n} is Cohen–Macaulay in precisely those characteristics relatively prime to n. Moreover, 7.1 gives us a significant bound on the depth when this ring is not Cohen–Macaulay: depth $R^{C_n} \leq 2 + n/p_0$, p_0 being the smallest prime dividing n, as compared with $\dim R^{C_n} = n - 1$.

PART II: INVARIANT RINGS BY ALTERNATING GROUPS THAT ARE NOT F-RATIONAL

In Part II we address an instance of the following question raised in the introduction to this paper.

QUESTION 2. *Let G be a finite group and fix an embedding of G as a subgroup of a permutation group S_n. This determines a k-algebra action of G on $k[X_1, \ldots, X_n]$ via $\sigma(X_i) := X_{\sigma(i)}$, $\sigma \in G$. For which choices of n, $G \le S_n$, and k is $k[X_1, \ldots, X_n]^G$ F-rational?*

Recall from 2.2 that this question had previously been answered when char(k) is relatively prime to $|G|$: In this case $k[X_1, \ldots, X_n]^G$ is F-rational. This was essentially the only progress on this problem, as we explained in the introduction to this paper.

The instance we address is when G is the alternating group A_n embedded in S_n in the usual way (so that A_n is the subgroup of S_n consisting of the even permutations).

We show in Theorem 12.2 that the resulting invariant ring $k[X_1, \ldots, X_n]^{A_n}$ is not F-rational for a certain class of n and $p = \text{char}(k)$. Indeed, in Theorem 12.3 we show that their non-F-rational loci have positive dimension, so that these rings do not merely fail to be F-rational at their localizations at some maximal ideals.

These results are somewhat surprising because the rings $k[X_1, \ldots, X_n]^{A_n}$ possess the major properties of graded F-rational rings. Namely, we know from Sections 1 and 2 that $k[X_1, \ldots, X_n]^{A_n}$ is a normal Cohen–Macaulay F-injective (even F-pure) domain with a negative a-invariant.

Admittedly, the line of argument through which we obtain the results of Part II is not as interesting as the line of argument in Part I. The argument in Part I used powerful results (the normal basis theorem of 1.3 and the spectral sequences of 1.1) with which we related the Cohen–Macaulay issue to an entirely different issue—the size of the étale locus of $k[X_1, \ldots, X_n]^G \subseteq k[X_1, \ldots, X_n]$. Furthermore, one result (Theorem 4.1) might apply to further studies of Question 1 not covered in this paper, as we explained at the end of Section 7.

In contrast, the line of argument in Part II is much more specific to the particular rings under consideration and brings in less interesting concepts. For in 9.1, the issue of F-rationality in $k[X_1, \ldots, X_n]^{A_n}$ is reduced to an ideal membership problem in the ring, which we then solve for certain values of n and char(k). The ideas we develop en route do not seem to be applicable to studying Question 2 for $G \neq A_n$.

Given this, why is it interesting to look at this example? We see two reasons.

One reason is that it adds to the collection of examples that fail to be F-rational but not for any obvious reason. That is, as we mentioned above, the rings $k[X_1, \ldots, X_n]^{A_n}$ have all the major characteristics of a graded F-rational ring, so that their failure to be F-rational is not easily explained. Such a collection is obviously useful in the study of F-rationality. (See Section 8 for a completely different set of examples in this collection.)

The second reason is that it gives the first nontrivial answer (other than Proposition 2.2) to a basic question (Question 2) about tight closure. Eventually one would like to know what properties of n, $G \leq S_n$, and k make $k[X_1, \ldots, X_n]^G$ F-rational.

The main reduction in the problem occurs in Sections 9 and 10. In Section 9, we explain how the usually challenging task of deciding weak F-regularity becomes significantly more feasible in the rings $k[X_1, \ldots, X_n]^G$. (It becomes a direct summand issue: See Proposition 9.1.) This information will be useful to us because weak F-regularity coincides with F-rationality in Gorenstein rings such as $k[X_1, \ldots, X_n]^{A_n}$. It is through this information that we reduce the F-rationality to an ideal membership problem in 10.1.

Starting in Section 11, we focus on solving the reduced problem. Section 11 contains lemmas used to solve this problem in Theorem 12.1. The F-rationality is now (negatively) decided. We state these findings in Theorem 12.2. Theorem 12.3 is a sharpening of 12.2 in which the non-F-rational locus is shown to be of positive dimension.

8. A PREVIOUSLY KNOWN EXAMPLE

This section contains an example of a ring that is vastly different from the rings $k[X_1, \ldots, X_n]^{A_n}$ but has the same interesting combination of properties. Specifically, the ring R below is a normal graded F-pure Gorenstein domain with a negative a-invariant, but is not F-rational. Using techniques analogous to those in the proof of 12.3, one can deduce from this information that the non-F-rational locus of R must, furthermore, have positive dimension.

EXAMPLE. Consider the graded normal domain

$$R := \frac{k[X, Y, Z, W]}{(X^3 + Y^3 + Z^3)}.$$

R is easily checked to have a negative a-invariant in all characteristics and to be F-pure in characteristics p for which $p \equiv 1 \bmod 3$. (The a-invariant of a hypersurface is well known to be the degree of the equation minus the sum of the degrees of the variables. Using the criterion of [Fe], the

F-purity of this ring amounts to the easily verified claim that $(X^3 + Y^3 + Z^3)^{p-1} \notin (X^p, Y^p, Z^p)$ when $p \equiv 1 \bmod 3$.)

However, R cannot be F-rational, no matter what the characteristic is: The F-rationality of a Gorenstein ring passes to direct summands (see [HH1]), but the direct summand

$$\frac{k[X,Y,Z]}{(X^3 + Y^3 + Z^3)}$$

cannot be F-rational, since its a-invariant is zero (see 2.3). Thus, the ring R, considered in characteristics $p \equiv 1 \bmod 3$, has the desired properties.

One sees from this example how to construct similar rings with the same properties: Construct a hypersurface

$$H = \frac{k[X_1, \dots, X_n]}{(f)}$$

with $n \geq 3$ according to the following requirements on the polynomial f and on $p = \operatorname{char}(k)$.

1. f is homogeneous for some assignment of $\deg X_i > 0$ and $\deg f = \sum_i \deg X_i$. (This makes $a(H) = 0$.)

2. f has an isolated singularity. (This ensures normality because $n \geq 3$.)

3. $f^{p-1} \notin (X_1^p, \dots, X_n^p)$. (This ensures F-purity [Fe].)

A similar argument now shows that $R := H[W]$ is a graded normal F-pure Gorenstein domain with $a(R) < 0$ but that R is not F-rational.

9. TIGHT CLOSURE IN THE RINGS $k[X_1, \dots, X_n]^G$

In this section we consider the general problems of studying tight closure in the rings $k[X_1, \dots, X_n]^G$. Of course, we really have Question 2 in mind, so we will also relate this information to Question 2. Specifically, our synopsis of the general problem contains a restatement of Question 2 as an ideal membership problem in the case of Gorenstein rings $k[X_1, \dots, X_n]^G$. (This is in Proposition 9.1.) The rest of the synopsis is a summary of pertinent results from Sections 1 and 2.

There are some important advantages that the rings $k[X_1, \dots, X_n]^G$ have from the point of view of studying tight closure. Perhaps the most important is that the usually challenging task of deciding weak F-regularity becomes significantly more feasible in the rings $k[X_1, \dots, X_n]^G$. The issue is simply whether $k[X_1, \ldots, X_n]^G \subseteq k[X_1, \ldots, X_n]$ splits (9.1, part (2)). (Recall from Section 2 that a ring is weakly F-regular when all its ideals are tightly closed. The main properties of weak F-regularity used in this section are that it implies F-rationality and the two coincide in Gorenstein rings.) If $k[X_1, \ldots, X_n]^G$ is Gorenstein, the issue is yet more concrete, for it is simply an ideal membership problem (9.1, part (4)).

Another advantage is that the rings $k[X_1, \ldots, X_n]^G$ have many of the characteristics of graded F-rational or weakly F-regular rings. We list these properties in 9.1, parts (1) and (3).

Before continuing the discussion, we verify these assertions.

9.1. PROPOSITION. *Let R be the polynomial ring $k[X_1, \ldots, X_n]$ over a field k. Consider the k-algebra action of a finite group $G \leq S_n$ on R determined by $\sigma(X_i) = X_{\sigma(i)}$ for $\sigma \in G$. The ring of invariants has the following properties.*

(1) *R^G is a graded normal F-pure domain.*

(2) *R^G is weakly F-regular if and only if it is a direct summand of R as an R^G-module. In particular, if R^G is Gorenstein, then it is F-rational if and only if it is a direct summand of R.*

(3) *If R^G is Cohen–Macaulay, it has a negative a-invariant.*

(4) *Assume R^G is Gorenstein. Then R^G is F-rational if and only if it has a homogeneous system of parameters $y_1, \ldots, y_n$ with socle element u for which $u \notin (y_1, \ldots, y_n)R$.*

Proof. All follows immediately from Sections 1 and 2, except for the statement in part (4) that $u \notin (y_1, \ldots, y_n)R$ implies R^G is F-rational. This implication follows from a result in [HR2]: According to Proposition 5.8, p. 151 in [HR2] (which uses that R^G is Gorenstein and $R^G \subseteq R$ is a module-finite degree-preserving map of graded rings), to see that $R^G \subseteq R$ splits, we only need to check that

$$(y_1 \ldots y_n)^{t-1} u \notin (y_1^t, \ldots, y_n^t)R \qquad \text{for all } t \geq 1.$$

(In other words, recalling that $(y_1 \ldots y_n)^{t-1}u$ generates the socle for $y_1^t, \ldots, y_n^t$, this is saying we only need to check that the countably many ideals $(y_1^t, \ldots, y_n^t)$ are contracted from R.) But this follows from the hypothesis $u \notin (y_1, \ldots, y_n)$, because $y_1, \ldots, y_n$ is a regular sequence in R. (Indeed, $y_1, \ldots, y_n$ is a homogeneous system of parameters in R, because they are homogeneous in R and the ideal they generate in R has height

$\text{ht}(y_1,\dots,y_n)R = \text{ht}(y_1,\dots,y_n)R^G = n$.) That is, from the assumption that $(y_1 \cdots y_n)^{t-1}u \in (y_1^t,\dots,y_n^t)R$ for some $t \geq 1$, one deduces that $u \in (y_1,\dots,y_n)R$, contradicting our hypotheses.

So here we have a class of rings (the rings $k[X_1,\dots,X_n]^G$) that are good candidates for weakly F-regular rings, and more importantly, their weak F-regularity is decided by the following natural question.

9.2. QUESTION. *What properties of n, $G \leq S_n$, and k make $k[X_1,\dots,X_n]^G \subseteq k[X_1,\dots,X_n]$ split?*

As with Question 2, there were essentially no results on this with the exception of 2.2. That is, prior to this paper one could only give trivial answers to 9.2 in characteristics dividing $|G|$. (Namely, Example 1 in Section 3 splits because $k[X_1,\dots,X_n]^{S_n}$ is regular (hence weakly F-regular). Example 4 in Section 3 does not split in characteristic p because $k[X_1,\dots,X_{p^e}]^{C_{p^e}}$ is not Cohen–Macaulay (hence not weakly F-regular).)

Of course, Question 1 points out a shortcoming of the rings $k[X_1,\dots,X_n]^G$—that there are relatively few instances in which we know whether $k[X_1,\dots,X_n]^G$ is Cohen–Macaulay, a precondition for F-rationality. However, this need not be disadvantageous to our study, for it may be easier to decide if $k[X_1,\dots,X_n]^G \subseteq k[X_1,\dots,X_n]$ splits than to decide if $k[X_1,\dots,X_n]^G$ is Cohen–Macaulay. (So, in a way, the shortcoming provides an added incentive: If one can show the extension $k[X_1,\dots,X_n]^G \subseteq k[X_1,\dots,X_n]$ to split in a particular instance, then one has also shown the ring $k[X_1,\dots,X_n]^G$ to be Cohen–Macaulay.)

This concludes our general discussion of tight closure in $k[X_1,\dots,X_n]^G$. We next focus on the case in which $G = A_n$.

10. NOTATION AND PRELIMINARY REMARKS ON THE F-RATIONALITY OF $k[X_1,\dots,X_n]^{A_n}$

We now begin the attack on deciding the F-rationality of $k[X_1,\dots,X_n]^{A_n}$. In this section we formulate a specific ideal membership problem (10.1) in this ring that is equivalent to deciding F-rationality. We also set up some parameters and notation that will be used in subsequent sections to answer 10.1. We will only be able to answer 10.1 for certain values of n and char(k).

Our goal is to decide whether $k[X_1,\dots,X_n]^{A_n}$ is F-rational. Of course, the action of A_n that we are considering is the variable permuting k-action under which $\sigma(X_i) = X_{\sigma(i)}$ for $\sigma \in A_n$.

First, we set up some restrictions on the parameters char(k), which we will denote by p and n. We assume n and p are at least three. Also, since

$k[X_1, \ldots, X_n]^{A_n}$ is already known to be F-rational in characteristics relatively prime to $|A_n| = n!/2$ (see 2.2), we assume $p = \mathrm{char}(k)$ falls within the range $3 \leq p \leq n$.

Next, we reduce the problem to one of ideal membership. Recall from Example 2 that $k[X_1, \ldots, X_n]^{A_n}$ is Gorenstein with homogeneous system of parameters $e_1, \ldots, e_n$ and socle element Δ (where, as in Example 2, e_i indicates the elementary symmetric polynomial of degree i in $X_1, \ldots, X_n$ and Δ is the discriminant $\prod_{i<j}(X_i - X_j)$). Thus, part (4) of Proposition 9.1 reduces deciding the F-rationality of $k[X_1, \ldots, X_n]^{A_n}$ to answering the following very concrete question:

10.1. QUESTION. *Is the element Δ in the ideal $(e_1, \ldots, e_n)k[X_1, \ldots, X_n]$?*

Explicitly, if the answer to 10.1 is "No" for a particular n and p, then the ring $k[X_1, \ldots, X_n]^{A_n}$ is F-rational for this n and $p = \mathrm{char}(k)$. If the answer is "Yes," then $k[X_1, \ldots, X_n]^{A_n}$ is not F-rational for this n and p.

We fix some notation that will be used in answering Question 10.1.

Notation for 11.1, 11.2, *and* 12.1. For $1 < k \leq n$ and $1 \leq i \leq k$, let

$$\Delta_k = \prod_{1 \leq i < j \leq k} (X_i - X_j)$$

$$\delta_k = \prod_{i=1}^{k-1} (X_i - X_k) = (X_1 - X_k)(X_2 - X_k) \cdots (X_{k-1} - X_k)$$

$e_i^{(k)}$ = the elementary symmetric polynomial of degree i in $X_1, \ldots, X_k$

$e_0^{(k)} = 1.$

11. LEMMAS CONCERNING THE ELEMENTARY SYMMETRIC FUNCTIONS

This section contains lemmas about the elementary symmetric functions that will be applied in Theorem 12.1. See the end of Section 10 for the notation used in this section.

The following easy lemma expresses the elementary symmetric functions in the variables $X_1, \ldots, X_k$ in terms of X_{k+1} and the elementary symmetric functions in $X_1, \ldots, X_{k+1}$.

11.1. LEMMA. *Fix i and k. Then $e_i^{(k)} = \sum_{j=0}^{i}(-1)^{i-j}X_{k+1}^{i-j}e_j^{(k+1)}$.*

Proof. Fixing k, the argument is a simple induction on i, noting that

$$e_{i+1}^{(k)} = e_{i+1}^{(k+1)} - X_{k+1}e_i^{(k)}$$

for $0 \leq i \leq n-1$.

We apply Lemma 11.1 to express the elementary symmetric polynomials in $n-1$ and $n-2$ variables in terms of those in n variables.

11.2. COROLLARY. *For fixed i and k, we have*

$$e_i^{(n-1)} = \sum_{j=0}^{i} (-1)^{i-j} X_n^{i-j} e_j^{(n)}$$

$$e_i^{(n-2)} = \sum_{j=0}^{i} \sum_{k=0}^{j} (-1)^{i-k} X_{n-1}^{i-j} X_n^{j-k} e_k^{(n)}.$$

Proof. These are immediate from Lemma 11.1.

12. THE F-RATIONALITY OF $k[X_1, \ldots, X_n]^{A_n}$

This section contains the main result of Part II, that $k[X_1, \ldots, X_n]^{A_n}$ is not F-rational in certain of the characteristics that divide $|A_n|$ (Theorem 12.2), and indeed fails to be F-rational in a significant way (Theorem 12.3). This result is preceded by a lemma (12.1) that solves an ideal membership problem (Question 10.1) to which the F-rationality issue was reduced in Section 10.

Lemma 12.1 answers Question 10.1 in certain characteristics.

12.1. LEMMA. *Consider the polynomial ring $k[X_1, \ldots, X_n]$ with $n \geq 3$ over a field k of positive characteristic p. Let $e_i \in k[X_1, \ldots, X_n]$, for $1 \leq i \leq n$, be the ith symmetric polynomial, in $X_1, \ldots, X_n$, and let $\Delta = \prod_{i<j}(X_i - X_j)$.*

Then $\Delta \in (e_1, \ldots, e_n)k[X_1, \ldots, X_n]$ if $n \equiv 0$ or $1 \bmod p$. (And actually, $\Delta \in (e_1, \ldots, e_{n-1})k[X_1, \ldots, X_n]$ if $n \equiv 0 \bmod p$.)

Proof. This proof uses notation (e.g., δ_n) established at the end of Section 10.

Using Corollary 11.2, and recalling that $e_i = e_i^{(n)}$, we rewrite δ_n and δ_{n-1} modulo $(e_1, \dots, e_n)$ as

$$\begin{aligned}
\delta_n = \prod_{i=1}^{n-1}(X_i - X_n) &= \sum_{i=0}^{n-1}(-1)^{n-1-i}X_n^{n-1-i}e_i^{(n-1)} \\
&= \sum_{i=0}^{n-1}\sum_{j=0}^{i}(-1)^{n-1-j}X_n^{n-1-j}e_j^{(n)} \\
&\equiv \sum_{i=0}^{n-1}(-1)^{n-1}X_n^{n-1} \bmod(e_1, \dots, e_{n-1}) \\
&= (-1)^{n-1}\sum_{i=0}^{n-1}X_n^{n-1} = (-1)^{n-1}nX_n^{n-1},
\end{aligned}$$

and, similarly,

$$\begin{aligned}
\delta_{n-1} = \prod_{i=1}^{n-2}(X_i - X_{n-1}) &= \sum_{i=0}^{n-2}(-1)^{n-2-i}X_{n-1}^{n-2-i}e_i^{(n-2)} \\
&= \sum_{i=0}^{n-2}\sum_{j=0}^{i}\sum_{k=0}^{j}(-1)^{n-2-k}X_{n-1}^{n-2-i}X_n^{j-k}e_k^{(n)} \\
&\equiv \sum_{i=0}^{n-2}\sum_{j=0}^{i}(-1)^{n-2}X_{n-1}^{n-2-i}X_n^{i} \bmod(e_1, \dots, e_{n-1}) \\
&= (-1)^{n-2}\sum_{i=0}^{n-2}(n-1-i)X_{n-1}^{n-2-i}X_n^{i} \\
&= (-1)^{n-2}(n-1)X_{n-1}^{n-2} + X_n \times \text{other terms} \\
&\in \left((n-1)X_{n-1}^{n-2}, X_n\right)\bmod(e_1, \dots, e_{n-1}).
\end{aligned}$$

Thus,

$$\begin{aligned}
\Delta = \Delta_n = \delta_n\delta_{n-1}\Delta_{n-2} &\in \left(nX_n^{n-1}\right)\left((n-1)X_{n-1}^{n-2}, X_n\right)\bmod(e_1, \dots, e_{n-1}) \\
&= \left(n(n-1)X_n^{n-1}X_{n-1}^{n-2}, nX_n^n\right)\bmod(e_1, \dots, e_{n-1}).
\end{aligned}$$

So $\Delta \in (e_1, \dots, e_{n-1})k[X_1, \dots, X_n]$ if $n \equiv 0 \bmod p$.

If $n \equiv 1 \bmod p$, then the element Δ is in $(e_1, \dots, e_n)k[X_1, \dots, X_n]$ because $X_n^n \in (e_1, \dots, e_n)k[X_1, \dots, X_n]$. ($X_n$ is a root of the polynomial $\prod_{i=1}^n(Z - X_i)$, whose coefficients are $\pm e_1, \dots, \pm e_n$.)

We can now decide the F-rationality of $k[X_1, \dots, X_n]^{A_n}$ for certain values of n and $\mathrm{char}(k)$.

12.2. THEOREM. *Let k be a field of characteristic $p \geq 3$. Let the alternating group A_n act on $k[X_1, \ldots, X_n]$ by permuting the variables. That is, consider the k-algebra action of A_n determined by $\sigma(X_i) := X_{\sigma(i)}$ for $\sigma \in A_n$.*

Then $k[X_1, \ldots, X_n]^{A_n}$ is an $\mathbf{N}$-graded Gorenstein F-pure normal domain with negative a-invariant for all n, but is not F-rational if $n \geq 3$ and $n \equiv 0$ or 1 mod p.

Proof. This follows from Proposition 9.1, Section 10 (specifically the reduction in Section 10 to Question 10.1), and Lemma 12.1, which answers Question 10.1 in the described characteristics.

We can immediately conclude from Theorem 12.2 and the graded F-rationality criterion in 2.5 that $k[X_1, \ldots, X_n]^{A_n}$ cannot have an isolated non-F-rational point at its homogeneous maximal ideal. The next theorem draws the yet stronger conclusion that $k[X_1, \ldots, X_n]^{A_n}$ cannot fail to be F-rational only at (finitely many) maximal ideals.

12.3. THEOREM. *Let k be a field of characteristic $p \geq 3$. Let the alternating group A_n act on the polynomial ring $R = k[X_1, \ldots, X_n]$ by permuting the variables. (See Theorem 12.2 for the explicit description of this action.) Then the locus in* Spec R^{A_n} *where R^{A_n} is not F-rational has positive dimension.*

Proof. Let I be the radical ideal defining the non-F-rational locus $V(I)$. (See Proposition 2.5.) Suppose $\dim V(I) = 0$. Then I is the intersection of a finite number of maximal ideals. But I is homogeneous (by 2.5), so any minimal prime of I is homogeneous. Thus I must be the homogeneous maximal ideal, contradicting the observation preceding this theorem that $k[X_1, \ldots, X_n]^{A_n}$ cannot have an isolated non-F-rational point.

Remark. While Theorem 12.3 tells us there is *some* nonmaximal prime P for which $k[X_1, \ldots, X_n]_P^{A_n}$ is not F-rational, we can give more specific information in the case $n \equiv 0 \bmod p$. From Lemma 12.1, we know Δ is in $(e_1, \ldots, e_{n-1})k[X_1, \ldots, X_n]$, instead of just in $(e_1, \ldots, e_n)k[X_1, \ldots, X_n]$. But $\Delta \notin (e_1, \ldots, e_{n-1})k[X_1, \ldots, X_n]_{e_n}^{A_n}$ because $e_1, \ldots, e_n$ is a regular sequence in $k[X_1, \ldots, X_n]^{A_n}$. From this and 2.6 we can conclude that $R_{e_n}^{A_n}$ is not F-rational: To be F-rational, $R_{e_n}^{A_n}$ would need to be a direct summand of R_{e_n}, but the ideal $(e_1, \ldots, e_{n-1})R_{e_n}^{A_n}$ is not contracted. Thus, there is a homogeneous nonmaximal prime P not containing e_n such that $R_P^{A_n}$ is not F-rational. (Specifically, we can take P to be any minimal prime of I not containing e_n, where I defines the non-F-rational locus.)

Our results on F-rationality in $k[X_1, \ldots, X_n]^{A_n}$ lead to a number of questions.

Questions. 1. Do the singular and non-F-rational loci of $k[X_1, \ldots, X_n]^{A_n}$ coincide when this ring is not F-rational (i.e., in characteristics in which this ring is not F-rational)?

The answer to this question would obviously be "No" if one omitted the restriction that $k[X_1, \ldots, X_n]^{A_n}$ not be F-rational, for this ring is never regular, but there do exist characteristics (e.g., those relatively prime to $n!/2$) in which it is F-rational.

But the question as stated makes sense: Fix a value of char(k) for which the ring $k[X_1, \ldots, X_n]^{A_n}$ is not F-rational. (Theorem 12.2 provides some such characteristics, e.g., char(k) dividing n.) Now ask: Does this ring $k[X_1, \ldots, X_n]^{A_n}$ only fail to be regular at primes where it fails to be F-rational?

There is some evidence for an affirmative answer: In the case $n = p = 3$, both loci are defined by the ideal (e_1, e_2, Δ).

We remark that, while there is no algorithm to compute the non-F-rational locus, the singular locus can be found in a straightforward manner. Recalling that $k[X_1, \ldots, X_n]^{A_n}$ is the hypersurface

$$\frac{k[e_1, \ldots, e_n, Z]}{(Z^2 - \Delta^2)},$$

we see that its singular locus is defined by Δ and the partial derivatives of Δ^2 with respect to each of the algebraically independent elements e_i. (We are assuming char(k) $\neq 2$.) Thus, we simply need to express the symmetric polynomial Δ^2 as a polynomial in the elementary symmetric functions $e_1, \ldots, e_n$. There are various algorithms for doing this. One way is to compute the resultant $R(f, f')$ of the polynomial

$$f(X) := X^n - e_1 X^{n-1} + \cdots + (-1)^n e_n$$

with its derivative f'. This computation, which simply involves taking the determinant of a matrix, produces a polynomial in $e_1, \ldots, e_n$. But $R(f, f')$ is well known to be the discriminant $\Delta^2 = \prod_{i \neq j}(X_i - X_j)$, i.e., the product of the differences of the roots of f.

2. Can one generalize Lemma 12.1 to other characteristics dividing $|A_n|$? If so, one could also generalize Theorems 12.2 and 12.3, to show the ring $k[X_1, \ldots, X_n]^{A_n}$ not to be F-rational in these characteristics. (It may even turn out that the ring $k[X_1, \ldots, X_n]^{A_n}$ is F-rational in *precisely* those characteristics relatively prime to $n!/2$. But this is only speculation.)

We, obviously, see no apparent generalization, even to the case $n \equiv 2 \bmod p$ and $n, p \geq 3$. But we did verify, using the program Macaulay, that the conclusion of 12.1 continues to hold for the smallest values of $n \geq 3$ and $p \geq 3$ to which it does not apply. That is, R^{A_5} is not F-rational in characteristic 3. (One checks that Δ is in $(e_1, \ldots, e_5)k[X_1, \ldots, X_5]$.)

13. CONCLUSIONS

In this paper, we have studied the related properties of Cohen–Macaulay and F-rationality in certain of a large class of invariant rings, specifically those obtained by the variable permuting action of a finite group $G \leq S_n$ on a polynomial ring $k[X_1, \dots, X_n]$.

Ultimately, one would like to classify the groups G and fields k for which the invariant ring $k[X_1, \dots, X_n]^G$ is Cohen–Macaulay (resp., F-rational). Some of the tools used in our analyses may be useful in this effort. Specifically, Theorem 4.1 could be useful in studying the Cohen–Macaulay property if one learns more about the locus where the action of G is not free. The results we obtain in Part II might be useful in completing an analysis of the characteristics in which $k[X_1, \dots, X_n]^{A_n}$ is F-rational.

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