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STRONG F-REGULARITY IN IMAGES OF REGULAR RINGS

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ABSTRACT. We characterize strong F-regularity, a property associated with *tight closure*, in a large class of rings. A special case of our results is a workable criterion in complete intersection rings.

Tight closure is a recently introduced operation linked to a variety of results in commutative algebra. Among the major results achieved with its use are a generalization of the Direct Summand Conjecture for rings containing a field [HH2] and a greatly simplified proof that invariant rings of regular rings by reductive groups are Cohen-Macaulay [HH1]. (Other applications are found in [Ho], [HH3], [HH4], and [S].) Hoping to discover additional results and to understand what makes tight closure so effective, one is led to examining tight closure itself. One way to do this is to study related concepts such as *strong F-regularity*.

The definition of strong F-regularity is not intuitively clear. An F-finite reduced ring R of prime characteristic p is *strongly F-regular* if for every element c of R that is not in any minimal prime of R, the R-linear map $R \to R^{1/q}$ sending $1 \mapsto c^{1/q}$ has an R-linear retraction for all sufficiently large powers q of p [HH3]. (Here, $R^{1/q}$ denotes the set of qth roots of the elements of R, regarded as an R-module in the natural way.)

While its definition may seem largely technical, this concept turns out to fit nicely with a number of ring properties. Strongly F-regular rings are between regular rings and Cohen-Macaulay normal rings [HH3]. They are always F-pure and have a negative a-invariant in the graded case ([HH3],[HH2]). And, of course, they're connected with tight closure: Every ideal in a strongly F-regular ring is *tightly closed* ([HH3]).

What we contribute to an understanding of strong F-regularity is a characterization (Theorems 2.3 and 3.1) in a large class of rings: A homomorphic image S/I of an F-finite regular local or F-finite regular graded ring S of characteristic p with (homogeneous) maximal ideal m, assumed to have infinite residue field in the graded case, by a (homogeneous) radical ideal I is strongly F-regular if and only if $s(I^{[p^e]} : I) \not\subseteq m^{[p^e]}$ for some $e \geq 1$, where s is a (homogeneous) element of S at

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which S/I is regular and not in any minimal prime of I. (See Section 1 for the definitions of $I^{[p^e]}$ and $m^{[p^e]}$.)

In other words, checking whether such a ring is strongly F-regular amounts to a familiar task (finding s at which S/I is regular) and a series of ideal containment problems.

These criteria are a good deal simpler in the complete intersection case, and often provide a reasonable way to test strong F-regularity in examples: See Section 4.

Independently of our work, N. Hara gave an elegant different proof for the special case of graded complete intersection rings [Ha]. We also acknowledge R. Fedder, whose paper [Fe] provided in large measure the inspiration for this paper. ([Fe] contains a similar characterization of F-purity.)

Theorem 2.3 in this paper is from the author's Ph.D. thesis. The author thanks her advisor Melvin Hochster for guidance, and thanks the referee for helpful comments.

The rings in this paper are tacitly assumed to be Noetherian of prime characteristic. This characteristic is denoted p, e denotes a positive integer, q denotes p^e, and R denotes a ring (Noetherian of characteristic p).

1. Background

Essential to the definitions used in this paper and to the theory of tight closure is the Frobenius endomorphism. This is the map $\mathbf{F} : R \to R$ that sends x to x^p. (Recall that R denotes a ring of characteristic p.) The eth iteration of this map, which sends x to x^q (recalling $q = p^e$), is denoted by $\mathbf{F}^e$.

One Frobenius-inspired notion is the "bracket power" of an ideal. If $I \subseteq R$ is an ideal, $I^{[q]}$ denotes the ideal generated by the qth powers of the elements of I. Note that $I^{[q]}$ is also generated by the qth powers of a set of generators for I.

Because $\mathbf{F}^e : R \to R$ is a ring homomorphism, it determines a restriction of scalars functor in the category of R-modules. We denote this functor by a pre-superscript e. Explicitly, if M is an R-module, then ${}^eM$ is the R-module whose elements and abelian group structure agree with those of M, and whose scalar multiplication is given by

$$r.u := r^q u$$

for $r \in R, u \in M$. Also, if $\phi : M \to N$ is an R-homomorphism, then ${}^e\phi : {}^eM \to {}^eN$ is the R-linear map that agrees elementwise with the map ϕ: ${}^e\phi(u) = \phi(u)$ for $u \in M$.

The R-module ${}^eR$ is regarded as a ring by defining its multiplication to be that of R. Thus, the action of $r \in R$ on $s \in {}^eR$ results in the same element of ${}^eR$ as does the product of r^q and s in ${}^eR$. To avoid confusing these two operations (namely, the R-module action on ${}^eR$ and ring multiplication in ${}^eR$), we use two different notations ($r.s$ versus rs), and often specify whether elements of R are to be thought of in R or in ${}^eR$.

The R-module ${}^eM$ is given an ${}^eR$-module structure as well: If $r \in {}^eR$ and $u \in {}^eM$, we define

$$r.u := ru.$$

In other words, this is the R-module structure on M if we identify ${}^eR$ with R as rings and ${}^eM$ with M as abelian groups.

As an illustration of these concepts, let I be an ideal in R, and consider the ideals ${}^eI$ and I^eR in ${}^eR$. ${}^eI$ is the ideal of ${}^eR$ consisting of the elements of I. In contrast, I^eR, being the expansion of the ideal I to the ring ${}^eR$, is ${}^e(I^{[q]})$. (That is, $I^eR = {}^e(I^{[q]})$ as ${}^eR$-modules.)

We next discuss two ring properties that concern the Frobenius map.

1.1. **Definition.** Let R denote a ring of prime characteristic.

(1) R is said to be *F-finite* if ${}^1R$ is a finitely-generated R-module.

(2) R is *F-pure* if $\mathbf{F} \otimes 1 : R \otimes_R M \to {}^1R \otimes_R M$ is injective for every R-module M.

It is easy to generate F-finite rings. Any localization of a finitely generated algebra over a perfect field is F-finite [Fe]. Localizations and homomorphic images of F-finite rings are F-finite [Fe].

F-purity is more subtle. Obviously, F-purity is a local issue, and F-pure rings are reduced. A quotient of a polynomial ring by an ideal generated by square-free monomials is F-pure [HR]. And, regular rings are F-pure [HR]. The proof of this last fact in the F-finite case also follows from the next proposition, which is essentially from [Ku1].

1.2. **Proposition.** *Let R be an F-finite regular local ring. Then ${}^1R$ is free over R. Consequently, ${}^eR$ is R-free for every positive integer e.*

Proof. $\operatorname{pd}_R {}^1R = \operatorname{depth} {}^1R - \operatorname{depth} R = 0$. Thus, ${}^1R$ is a projective and finitely generated module over the local ring R, so is R-free.

The remaining statement follows from induction on e. □

Proposition 1.2 has a converse: If ${}^eR$ is R-free for some e, then R is regular [Ku1].

Theorem 1.3 is a very useful characterization of F-purity. With it, for instance, the F-purity of a local hypersurface $S/(f)$ is decided by an ideal membership problem (specifically, in the notation of 1.3, whether f^{p-1} is in $m^{[p]}$).

1.3. **Theorem.** *Let S be a regular local ring with maximal ideal m, and let $R = S/I$. Then R is F-pure if and only if $I^{[p]} : I \not\subseteq m^{[p]}$.*

Proof. This is the main result (Theorem 1.12) in [Fe]. □

We shall also need the following results (1.4 and 1.5) from [Fe]. Recall that if S is a ring and S^* is an S-algebra, then $\operatorname{Hom}_S(S^*, S)$ has an S^*-module structure given by $(s.\phi)(t) := \phi(st)$, where $s, t \in S^*$ and $\phi \in \operatorname{Hom}_S(S^*, S)$.

1.4. **Proposition.** *Let $S \subseteq S^*$ be Gorenstein local rings such that S^* is a finitely generated free S-module. Then $\operatorname{Hom}_S(S^*, S) \cong S^*$ as an S^*-module.*

Furthermore, if T is a generator for $\operatorname{Hom}_S(S^*, S)$ as an S^*-module, I, an ideal of S, and s an element in S^*, then the image of an ideal H under $sT : S^* \to S$ is contained in I if and only if $s \in (IS^* : H)$.

Proof. [Fe] Lemma 1.6. Note that sT refers to the S^*-action on $\operatorname{Hom}_S(S^*, S)$. □

Proposition 1.4 tells us that the elements of $S^* \cong \operatorname{Hom}_S(S^*, S)$ that induce well-defined homomorphisms $S^*/H \to S/I$ are those in the ideal $(IS^* : H)$. Furthermore, the ones among these that define the zero map $S^*/H \to S/I$ are those in IS^*. This explains the injectivity of ψ in the following corollary.

1.5. **Corollary.** *Under the hypotheses and notation of Proposition* 1.4*, there is an S-module isomorphism*

$$\psi : \frac{(IS^* : H)}{IS^*} \to \mathrm{Hom}_S\left(\frac{S^*}{H}, \frac{S}{I}\right)$$

given by

$$\psi(x + IS^*) := \overline{xT},$$

where $\overline{xT}$ is defined by

$$\overline{xT}\ \ (y + H) := (xT)(y) + I.$$

(*In other words, $\overline{xT}\ \ (y + H) = T(xy) + I$.*)

Proof. [Fe], Corollary to Lemma 1.6. □

Next, we cite pertinent facts concerning strong F-regularity. The following definition is equivalent to that given in the introduction.

1.6. **Definition.** Let R be an F-finite reduced ring (of characteristic p). We say that R is *strongly F-regular* if for every element c of R that is not in any minimal prime of R, the R-linear map $R \to {}^eR$ defined by $1 \mapsto c$ has an R-linear retraction for all sufficiently large e (equivalently for some e) [HH3].

Notation. We denote the R-linear map $R \to {}^eR$ sending 1 to c (i.e. the map $r \mapsto r^{p^e}c$) in the definition by $c\mathbf{F}^e$. Thus, $c\mathbf{F}^e$ is the composition

$$R \xrightarrow{\mathbf{F}^e} {}^eR \xrightarrow{c} {}^eR,$$

where the second map is multiplication by c thought of as an element of ${}^eR$.

Definition 1.6 is equivalent to the definition of strong F-regularity given in the introduction: The R-linear maps $c\mathbf{F}^e : R \to {}^eR$ and $f : R \to R^{1/q}$ via $f(1) = c^{1/q}$ are isomorphic. Consequently, the map $c\mathbf{F}^e : R \to {}^eR$ splits if and only if $f : R \to R^{1/q}$ does.

The next proposition makes the definition of strong F-regularity significantly more feasible to apply. It says that one only has to verify the splitting for some e of $c\mathbf{F}^e$ for one particular c.

1.7. **Proposition.** *Let R be an F-finite reduced ring.*

(1) *Let c be an element of R not in any minimal prime of R. Suppose there exists $e' \geq 1$ such that $c\mathbf{F}^{e'} : R \to {}^{e'}R$ has an R-linear retraction. Then $c\mathbf{F}^e$ has an R-linear retraction for all $e \geq e'$.*

(2) *Let s be an element of R not in any minimal prime of R such that R_s is regular* (*or even just strongly F-regular: such elements exist by* 1.8). *Then R is strongly F-regular if and only if there exists $e \geq 1$ for which $s\mathbf{F}^e : R \to {}^eR$ has an R-linear retraction.*

Proof. See Remark 5.4(d) and Theorem 5.9 in [HH3]. □

1.8. *Remark.* Elements s as in Proposition 1.7 do exist: See Remark 5.7 of [HH3]. In the case in which R is graded (i.e. $\mathbb{N}$-graded and finitely generated over a field k such that $k = [R]_0$), s may be taken to be homogeneous, since the radical ideal defining R's singular locus will be homogenous.

The following theorem relates strong F-regularity to a number of standard properties.

1.9. **Theorem.** *Let R be an F-finite reduced ring.*
(1) *If R is regular, then R is strongly F-regular.*
(2) *If R is strongly F-regular, then R is normal, Cohen-Macaulay, and F-pure.*
(3) *If R is strongly F-regular, then every ideal of R is tightly closed.*

Proof. See Theorem 5.5 of [HH3], Theorems 2.1 and 3.1 of [HH4], Remark 1.6 of [FW], Proposition 3.8 of [HH2], and Theorem 2.5 of [Ku2]. $\square$

2. A criterion for strong F-regularity

In this section we prove a criterion (Theorem 2.3) for strong F-regularity in images of F-finite regular local rings. Section 3 contains a graded version of this result, and Section 4 contains further discussion for complete intersection rings.

The strong F-regularity of a ring R is a question of whether certain R-linear maps $R \to {}^eR$ have R-linear retractions. Thus, in studying strong F-regularity, one is led to analyzing the modules $\mathrm{Hom}_R({}^eR, R)$. We do such an analysis in 2.1, leading to our main results in 2.3 and 3.1.

Our arguments in this section largely parallel ones in [Fe], in which a characterization of F-purity is found by a similar analysis of $\mathrm{Hom}_R({}^1R, R)$.

2.1. **Lemma.** *Let S be an F-finite regular local ring, and let R denote S/I for some ideal I. Then*

$$\mathrm{Hom}_R({}^eR, R) \cong {}^e\left(\frac{(I^{[q]} : I)}{I^{[q]}}\right)$$

as R-modules.

Proof. From Proposition 1.2 we know that ${}^eS$ is a free S-module. Thus we can apply 1.5 with $S^* := {}^eS$, $H := {}^eI$, and $I := I$ to obtain

$$\mathrm{Hom}_S({}^eR, R) \cong \frac{{}^eI^{[q]} :_{{}^eS} {}^eI}{{}^eI^{[q]}}$$
$$\cong {}^e\left(\frac{I^{[q]} :_S I}{I^{[q]}}\right)$$

as S-modules. (${}^eR \cong {}^eS/{}^eI$ because restriction of scalars is exact.) The result now follows because R-linearity coincides with S-linearity for maps ${}^eR \to R$. $\square$

2.2. **Lemma.** *Let S be an F-finite regular local ring with maximal ideal m. Let $\pi : S \to R$ be a ring surjection onto a ring R. Let I be the kernel of π. Fix $e \geq 1$ and $c \in S$. The map $\pi(c)\mathbf{F}^e : R \to {}^eR$ has an R-linear retraction if and only if $c \notin (m^{[q]} :_S (I^{[q]} :_S I))$.*

Proof. First some notation. If $\phi : {}^eS \to S$ is S-linear and maps ${}^eI$ to I, we will denote by $\pi(\phi)$ the corresponding R-linear map from ${}^eR$ to R (i.e. the one that sends $y + {}^eI$ to $\pi(\phi(y))$). (Recall that ${}^eR \cong {}^eS/{}^eI$ as R-modules.)

Let T be a generator of the ${}^eS$-module $\mathrm{Hom}_S({}^eS, S) \cong {}^eS$. (See Proposition 1.4.) By Corollary 1.5, the elements of $\mathrm{Hom}_R({}^eR, R)$ are of the form $\pi(sT)$ with $s \in {}^e(I^{[q]} : I)$. Thus, a retraction for $\pi(c)\mathbf{F}^e$ would be a map $\pi(sT)$ with $s \in {}^e(I^{[q]} : I)$ for which $\pi(sT)\pi(c)\mathbf{F}^e$ is the identity map on R, i.e. fixes the identity element 1_R of R. But $\pi(sT)\pi(c)\mathbf{F}^e(1_R) = \pi(T(sc))$. So $\pi(c)\mathbf{F}^e$ splits (has an R-linear retraction) precisely when the ideal $\pi(T(c(I^{[q]}:I)))$ is not proper. In other words, $\pi(c)\mathbf{F}^e$ splits if and only if

$$\pi(T(c(I^{[q]}:I))) \not\subseteq \pi(m),$$

i.e. when

$$T(c(I^{[q]}:I)) \not\subseteq m + I = m.$$

Since $T(c(I^{[q]}:I)) = (cT)(I^{[q]}:I)$, we can utilize 1.4 again to characterize the $c \in S$ for which $T(c(I^{[q]}:I)) \subseteq m$ as the c in $(m^{[q]}:(I^{[q]}:I))$, giving the result. □

Following is the main result of this section.

2.3. **Theorem.** *Let S be an F-finite regular local ring of prime characteristic p with maximal ideal m. Let R denote S/I for some proper radical ideal I. Let s be an element of S not in any minimal prime of I such that R_s is regular (or even just strongly F-regular: such elements exist by* 1.8.) *The following are equivalent.*
(1) *R is strongly F-regular.*
(2) *For each element c of S not in any minimal prime of I, $c(I^{[p^e]}:I) \not\subseteq m^{[p^e]}$ for all sufficiently large positive integers e.*
(3) *I is prime and $I = \bigcap_{e\geq 1}(m^{[p^e]}:(I^{[p^e]}:I))$.*
(4) *There exists a positive integer e such that $s(I^{[p^e]}:I) \not\subseteq m^{[p^e]}$.*

Proof. (1)≡(2): This follows directly from Lemma 2.2 and the definition of strong F-regularity.

(2)⟹(3): Assume R is strongly F-regular. Then I is prime because R is normal (see 1.9) and local. Then $I \supseteq \bigcap_e(m^{[p^e]}:(I^{[p^e]}:I))$ follows from (2), and $\subseteq$ holds in general.

(3)⟹(4): Obvious.

(4)⟹(1): 1.7 and 2.2. □

Note the similarity between Theorems 2.3 and 1.3, Fedder's F-purity criterion (1.3): Each expresses a Frobenius-related property of S/I to a property of bracket powers of I and m.

3. A graded version

This section contains a criterion (Theorem 3.1) for strong F-regularity in images of F-finite regular graded rings over infinite fields. By "graded", we mean $\mathbb{N}$-graded and finitely generated over a field k such that k consists of the elements of degree zero. Note that the F-finite regular graded rings are simply the polynomial rings $k[X_1,\ldots,X_n]$ over F-finite fields k.

3.1. **Theorem.** *Let R be a reduced $\mathbb{N}$-graded ring of prime characteristic p such that $[R]_0$ is an infinite F-finite field and R is a finitely generated $[R]_0$-algebra. Write*

$$R \cong \frac{k[X_1,\ldots,X_n]}{I}$$

where $k=[R]_0$, $k[X_1,\ldots,X_n]$ is an $\mathbb{N}$-graded polynomial ring over k, such that $k=[k[X_1,\ldots,X_n]]_0$ and $X_1,\ldots,X_n$ are homogeneous, and I is homogeneous.

Let s be a homogenous element of $k[X_1, \dots, X_n]$ not in any minimal prime of I for which R_s is regular or even just strongly F-regular. (*Such s exist by* 1.8.) *Let m be $(X_1, \dots, X_n)$. The following are equivalent.*

(1) *R is strongly F-regular.*

(2) *I is prime and $I = \bigcap_{e \geq 1}(m^{[p^e]} : (I^{[p^e]} : I))$.*

(3) *There exists a positive integer e such that $s(I^{[p^e]} : I) \not\subseteq m^{[p^e]}$.*

Proof. (1)$\Longrightarrow$(2): Assume R is strongly F-regular. Then R, being a graded normal (1.9) ring, is a domain, and so I is prime.

Let q denote p^e. Strong F-regularity localizes ([HH3]), so we can apply 2.3 to the local ring R_m to see that

$$I_m = \bigcap_q (m_m^{[q]} : (I_m^{[q]} : I_m)).$$

Set $S := k[X_1, \dots, X_n]$ and $J_q := m^{[q]} : (I^{[q]} : I)$. Note that each J_q is homogeneous and that $J_q S_m = m^{[q]} S_m : (I^{[q]} S_m : IS_m)$ because the colons commute with flat base change.

Thus, $I_m = \bigcap (J_q)_m$. Contracting each side back to S (i.e. applying $\bigcap S$) gives that $I = \bigcap_q (m^{[q]} : (I^{[q]} : I))$. (The ideals I and J_q are contracted with respect to $S \subseteq S_m$ because these ideals are homogeneous.)

(2)$\Longrightarrow$(3): Obvious.

(3)$\Longrightarrow$(1): For such an e, we will have $s(I_m^{[p^e]} : I_m) \not\subseteq m_m^{[p^e]}$ (by the homogeneity of $I^{[p^e]} : I, m^{[p^e]}$, and s). Applying 2.2 and 1.7, we then see that R_m is strongly F-regular.

But the radical ideal defining the complement of the strongly F-regular locus of R is homogeneous when k is infinite. (The strongly F-regular locus is open by [HH3], Theorem 5.9. One sees from this by applying standard techniques, i.e. the techniques of Discussion 4.1 and Theorem 4.2 in [HH2], that the radical ideal defining the complement of this locus is homogenous when k is infinite.) So R_m being strongly F-regular implies that R is strongly F-regular. $\square$

4. Special case: Complete intersections

The main result (Theorem 4.1) of this section is a workable criterion of strong F-regularity in complete intersection rings. The essence of case (2) of Theorem 4.1 was proven independently by N. Hara and appears in [Ha].

4.1. **Theorem.** *Let $S, m, G_1, \dots, G_d$ be as in one of the following two scenarios.*

(1) *S is an F-finite regular local ring of prime characteristic p, m is its maximal ideal, and $G_1, \dots, G_d$ is a proper regular sequence generating a radical ideal.*

(2) *$S = k[X_1, \dots, X_n]$ is an $\mathbb{N}$-graded polynomial ring over an infinite F-finite field k of prime characteristic p such that $k = [S]_0$ and $X_1, \dots, X_n$ are homogeneous, m is $(X_1, \dots, X_n)$, and $G_1, \dots, G_d$ is a proper homogeneous regular sequence generating a radical ideal.*

Let R be $S/(G_1, \dots, G_d)$. Let s be an element of S not in any minimal prime of $(G_1, \dots, G_d)$ for which R_s is regular (*or strongly F-regular*)*, choosing s to be homogeneous if R is graded.* (*Such elements s exist by Remark* 1.8.)

R is strongly F-regular if and only if there exists a positive integer e such that $s(\prod_{i=1}^{d} G_i)^{p^e-1} \notin m^{[p^e]}$.

Proof. This is an immediate corollary of Theorems 2.3 and 3.1, noting that if $I = (G_1, \dots, G_d)$, then $I^{[p^e]} : I = (G_1^{p^e}, \dots, G_d^{p^e}, \prod_{i=1}^{d} G_i^{p^e-1})$. $\square$

Example. The ring
$$R = \frac{k[X,Y,Z]_{(X,Y,Z)}}{(X^2+Y^2+Z^2+XYZ)},$$
where k is an F-finite field of characteristic not equal to two, is strongly F-regular.

Proof. One sees that for $s := 2Y + XZ$, $g := X^2+Y^2+Z^2+XYZ$, and $p := \mathrm{char}(k)$ that $sg^{p-1} \notin (X^p, Y^p, Z^p)_{(X,Y,Z)}$. (One way is to notice that sg^{p-1} has a monomial term $\mu \notin (X^p, Y^p, Z^p)$ with the degree of μ being minimal among the degrees of the monomials in sg^{p-1}.) $\square$

We thank the referee for pointing out the following corollary, reminiscent of the well-known fact that if S is a regular local ring with maximal ideal m and $f \in m^2$, then $S/(f)$ is not regular.

4.2. **Corollary.** *Let $S, m, G_1, \dots, G_d$, and R be as in 4.1, and let $n := \dim S$. If $(G_1, \dots, G_d) \neq m$ and $\prod_{i=1}^{d} G_i \in m^n$, then R is not strongly F-regular.*

Proof. Because $(G_1, \dots, G_d) \neq m$, there exists an element s, homogeneous in the graded case, of m and not in any minimal prime of $(G_1, \dots, G_d)$ for which R_s is regular. Let e be a positive integer. Then $s\prod_{i=1}^{d} G_i^{p^e-1} \in m^{n(p^e-1)+1} \subseteq m^{[p^e]}$, the latter inclusion being because m can be generated by n elements. $\square$

For example, $k[X,Y,Z]_{(X,Y,Z)}/(X^3+Y^3+Z^3+X^2YZ)$ is not strongly F-regular.

5. A question

To have a really useful strong F-regularity criterion, one would like an affirmative answer to the following.

Question. Given R and s as in Theorem 2.3 or 3.1, can we effectively determine an integer $E = E(s,R)$ such that
$$R \text{ is strongly F-regular } \iff s(I^{[p^E]} : I) \not\subseteq m^{[p^E]}?$$

In the case of complete intersection rings R, an affirmative answer would reduce testing strong F-regularity to determining s and E and deciding a single ideal membership problem.

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