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Sparse Systems of Parameters for Determinantal Varieties

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1. INTRODUCTION

Let $X \subset \mathbb{P}(V)$ be an irreducible projective algebraic variety, where V is a vector space of dimension N over an infinite field K. Most linear subspaces $\mathbb{P}(W) \subset \mathbb{P}(V)$ of codimension $\dim X + 1$ are disjoint from X. Such linear subspaces, whose defining equations are called *systems of parameters* for the coordinate ring of X, are important from both the computational and theoretical points of view; see [5, 11, 13] and their references. For instance, they can be used to compute cohomology of the coherent sheaves $\mathcal{O}_X(n)$ on X.

From a computational point of view, it is most convenient and efficient to work with a description of $\mathbb{P}(W)$ in terms of *sparse data*. For example, fixing homogeneous coordinates X_i for $\mathbb{P}(V)$, $\mathbb{P}(W)$ can be described as the common vanishing set of a collection of linear functionals $Y_i = \sum_{j=1}^{N} \lambda_{ij} X_j$ on V. These data are *sparse* if many of the coefficients λ_{ij} are zero. The minimal number of nonzero λ_{ij} that are required as we range over all linear systems of parameters $\{Y_i\}$ for X is the *Noether complexity*—a measure of how complex X is with respect to the chosen coordinates. Introduced by Eisenbud and Sturmfels [5], the Noether complexity is most interesting from the point of view of computational algebraic geometry, combinatorics, or coding theory, where data are usually presented in terms of a fixed and immutable choice of coordinates, However, even from a

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theoretical point of view, this number is interesting for varieties that come equipped with a natural set of coordinates; see, for example, [10].

This paper investigates *sparse systems of parameters for determinantal varieties*. Determinantal varieties have a preferred choice of coordinates; their rich combinatorial structure and important role throughout mathematics makes them an especially interesting example. We describe systems of parameters for determinantal rings that are both highly symmetric and which are sparse. We give formulas for the complexity of certain determinantal varieties for several variants of the notion of Noether complexity.

An auxiliary investigation for monomial rings was necessary in this study. We describe a nice combinatorial criterion for systems of parameters for monomial rings in Section 3. As pointed out by Eisenbud and Sturmfels, the Noether complexity of a projective variety is bounded above by the Noether complexity of the initial ideal (with respect to any term order) of its defining ideal. Our work indicates that determinantal varieties are "maximally complex" in the sense that the Noether complexity is actually equal to the upper bound provided by the complexity of the initial ideal, though we are able to prove this only for certain cases.

In the course of our investigation we discovered a variety Z such that *every* linear space disjoint from Z has *maximal complexity* with respect to any of the four variants of complexity introduced in [5]. That is, all linear spaces of maximal dimension disjoint from Z have the same complexity, and this is equal to the maximal possible complexity of an arbitrary linear space of that dimension. This is so regardless of the variant of complexity we use; see Section 4.

A final, more theoretical, reason to study the Noether complexity of projective varieties in general is to gain information about the Chow form. The linear spaces of codimension $d = \dim X + 1$ which intersect $X \subset \mathbb{P}(V)$ nontrivially form a hypersurface in the Grassmannian $\mathbb{G}\mathrm{r}^d(V)$ of codimension d subspaces of V. This hypersurface constitutes the point corresponding to X on the Chow variety of $d-1$-dimensional subvarieties in $\mathbb{P}(V)$. The Chow form is the equation, in Plücker coordinates, of this hypersurface. Eisenbud and Sturmfels pointed out that the Noether complexity can be "read off" the Chow form [5, 2.7]. In Section 3, we record the Chow form for monomial varieties. In practice, however, Chow forms are notoriously difficult to compute, and there is no formula known for the Chow form of determinantal varieties. For the case of maximal minors, however, the Chow form can be expressed as an $m \times n \times (n - m - 1)$ "hyperdeterminant"; see [7, 4.13]. Our study of systems of parameters for determinantal varieties is partially motivated by this connection with the Chow form.

2. SPARSE DATA AND COMPLEXITY

Let W be a codimension d linear subspace of a vector space V. We explain what we mean by "describing W by sparse data."

Sparsity notions are not associated to the abstract vector space V, but rather to V together with a fixed choice of basis, $\{e_i\} = e_1, \dots, e_n$. Our point of view will be that this basis for V is fixed and immutable. The elements of the dual basis $\{X_i\} \subset V^*$ will be called the "coordinate functionals" or "coordinates" for V:

$$X_i(e_j) = \begin{cases} 0, & \text{for } i \neq j, \\ 1, & \text{for } i = j. \end{cases}$$

A codimension d linear subspace $W \subset V$ can be represented in (at least) four different ways in terms of this fixed basis for V. A *basis representation* for W is a choice of basis for W, written out as a set of $N - d$ linear combinations of the elements e_i. A *cobasis representation* for W is a choice of d linear functionals on W whose common kernel is exactly W, written out as a set of d linear combinations of the coordinate functionals x_i. A Plücker basis representation is a choice of a basis for $\bigwedge^{N-d} W$ in $\bigwedge^{N-d} V$, written as a combination of the preferred basis elements $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{N-d}}$. The Plücker cobasis representation of W is defined dually, in terms of the basis $X_{i_1} \wedge X_{i_2} \wedge \cdots \wedge X_{i_d}$.

Note that while there is considerable choice in choosing basis and cobasis representations for W, both its Plücker basis and Plücker cobasis representations are determined by $\{e_i\}$ up to scalar multiple.

2.1

We will lose no generality by making this even more explicit with the following identifications. We identify V with the space spanned by the column vectors e_i consisting of zeros in each row except the ith row, where the entry is 1. The codimension d subspace W is represented by the maps

$$W \overset{A}{\hookrightarrow} V \overset{B}{\to} V/W.$$

That is, W is the column space of the $N \times (N - d)$ matrix A; it is also the kernel of the $d \times N$ matrix B. The rows of B can be thought of as linear functionals $\sum_j b_{ij} x_j$ on V. The choice of A and B are the choice of a basis and cobasis representation for W. The matrix A (respectively, B) can be altered by invertible column (respectively, row) operations to produce a new basis (respectively, cobasis) representation for W. The $N-d$ minors of A are a Plücker basis representation of W, while the d minors of B are a Plücker cobasis representation of W.

2.1. DEFINITION. The complexity of a matrix M with entries in K is the number of nonzero entries of M. The basis complexity with respect to the basis $\{e_i\}$ for V is the complexity of the least complex basis representation of W. The cobasis complexity of W with respect to the basis $\{e_i\}$ for V is the complexity of the least complex cobasis representation of W.

That is, the basis complexity is the complexity of the least complex matrix A such that W is the column space of A. The cobasis complexity is the complexity of the least complex matrix B such that W is the kernel of B. The Plücker basis complexity is the number of nonzero $N-d$ minors of any basis representing matrix A. The Plücker cobasis complexity is the number of nonzero d minors of any cobasis representing matrix B.

We record some easy general bounds on complexity for future reference.

2.2. PROPOSITION. *Let W be a codimension d subspace of an N-dimensional vector space V. Then, with respect to any basis for V, there are the following bounds on the complexity of W*:

(1) *The basis complexity of W is between $N-d$ and $(d+1)(N-d)$, inclusive.*

(2) *The cobasis complexity of W is between d and $(d)(N-d+1)$, inclusive.*

(3) *The Plücker basis and cobasis complexity are both between* 1 *and* $\binom{N}{d}$, *inclusive.*

Proof. The space W can be identified with the column space of an $N \times (N-d)$ matrix. All such matrices differ by column operations, that is, by the action of $\mathrm{GL}(N-d)$ on the right. Because the matrix is full rank, we may multiply by some element in $\mathrm{GL}(N-d)$ so as to create an $N-d \times N-d$ identity matrix inside some such matrix representing W. The bounds in (1) follow immediately.

The bounds in (2) follow similarly, since a cobasis representation of W is a full rank $d \times N$ matrix, up to the action of $\mathrm{GL}(d)$ on the left. The bounds in item (3) follow by computing the maximal minors in each of these extreme cases. ∎

We now turn our attention to the specific instance arising in computational algebraic geometry: the linear subspaces disjoint from a projective variety.

Let $X \subset \mathbb{P}(V)$ be a projective variety of dimension $d - 1$. Let $W \subset V$ be a subvector space of codimension d such that $\mathbb{P}(W) \cap X$ is empty (a sufficiently general choice of W will have this property). Let $R = K[X]$ be the homogeneous coordinate ring for X; it is a graded ring, with fixed presentation

$$R = \frac{K[X_1, X_2, \dots, X_N]}{I},$$

where I is the homogeneous ideal of relations on the coordinate functionals X_i. If K is an infinite field, then R has a system of parameters consisting of homogeneous elements of degree 1. These are d (= dimension R) elements $Y_1, Y_2, \dots, Y_d$ of the form

$$Y_i = \lambda_{i1} X_1 + \lambda_{i2} X_2 + \cdots + \lambda_{iN} X_N$$

that generate an ideal of R whose radical is $(X_1, \dots, X_N)$. Equivalently, a (linear) system of parameters $\{Y_i\}$ is a collection of d linear forms such that the corresponding hyperplanes of zeros intersect the affine cone over X only at the origin of V; that is, a system of parameters is precisely the same as a cobasis for a linear space of $\mathbb{P}(V)$ of maximal dimension of disjoint from X.

2.2. *Parameter Matrices*

The system of parameters will be represented by a $d \times N$ matrix, whose ith row is the N vector

$$(\lambda_{i1}, \lambda_{i2}, \dots, \lambda_{iN}),$$

giving the defining equation for Y_i. We call such a matrix a parameter matrix; it is simply the matrix B as in Section 2.1. Of course, not every $d \times N$ matrix defines a system of parameters for R—the parameter matrices are identified with a Zariski open subspace of affine dN space; see Section 3.2.

The Noether complexity of X or, equivalently, of R, with respect to the fixed coordinates X_i is defined as follows [5].

2.3. Definition. The Noether (cobasis) complexity of $X \subset \mathbb{P}(V)$ with respect to fixed coordinates X_i on V is the minimal possible cobasis complexity for a codimension d linear subvariety $\mathbb{P}(W) \subset \mathbb{P}(V)$ intersecting X trivially. Equivalently, it is the complexity of a sparsest possible parameter matrix for R.

Similarly, we define the Noether basis complexity and the Plücker basis and cobasis complexity. The Noether *cobasis* complexity is of primary interest in standard computations (see [5, 11]). We write $\mathrm{NC}(I)$ for the Noether cobasis complexity and refer to this as simply the "Noether complexity" without further qualification.

We stress that the definition of Noether complexity requires one to fix the coordinates $X_1, X_2, \ldots, X_N$, and then consider systems of parameters expressed in terms of these coordinates. Of course, one could always perform a linear change of coordinates so as to assume that $X_1', X_2', \ldots, X_d'$ are a system of parameters for $R \cong (K[X_1', X_2', \ldots, X_N'])/I$; that is, *there always exist coordinates such that the Noether complexity of I is d*. The point here is to work with a chosen set of coordinates.

Proposition 2.2 translates into some obvious bounds on Noether complexity.

2.4. GENERAL BOUND ON NOETHER COMPLEXITY. *Let I be any homogeneous ideal of* $K[X_1, X_2, \ldots, X_N]$. *The Noether basis complexity is no greater than* $(N-d)(d+1)$ *and no less than* $N-d$. *The Noether cobasis complexity of I is no greater than* $Nd - d(d-1)$ *and no less than* d. *The Noether Plücker basis and cobasis complexities are between* 1 *and* $\binom{N}{d}$, *inclusive*.

A parameter matrix for R remains a parameter matrix after multiplication on the left by any element of $\mathrm{GL}(d, K)$; the ideal of R generated by the corresponding linear functions is unchanged. This leads to the following relationship between the Noether basis and cobasis complexities. Essentially, we relate these complexities when some minimally complex parameter matrix can be "solved" without increasing its complexity.

2.5. PROPOSITION. *Let* $R = k[X_1, \ldots, X_N]/I$ *be a graded ring of dimension* d *and* $C = (\lambda_{ij})$ *be a parameter matrix for R of minimal complexity. Suppose that for some* $A \in \mathrm{GL}(d, k)$, AC *contains a* $d \times d$ *submatrix that is a permutation matrix and* AC *has no larger complexity than* C. *Then the Noether cobasis complexity of R is at least* $b - N + 2d$, *where b denotes the Noether basis complexity of R*.

Proof. Permuting rows and columns if necessary, we assume that the leftmost $d \times d$ minor of AC is the identity matrix. Write $AC = (\lambda_{ij})$ and let $cx(A)$ denote complexity of the matrix A. Associated to AC is a basis $v_1, \ldots, v_{N-d}$ for its null space of complexity $N - 2d + cx(AC)$, obtained by "back substitution" as follows. For each $j = 1, \ldots, N-d$, let v_j be the column vector that has a 1 in its $(j+d)$th entry, 0 in all other entries with indices $> d$, and whose first through dth entries are $-\lambda_{1,d+j}, \ldots, -\lambda_{d,d+j}$, respectively. All together this gives a basis of complexity $(N-d) + (cx(AC) - d) = N - 2d + cx(AC)$. Thus, $N - 2d + cx(AC) \geq b$, so $cx(C) \geq cx(AC) \geq b - N + 2d$. ∎

Example 7.13 shows that the inequality of Proposition 2.5 does not hold for general R.

An interesting question is for what, if any, varieties $X \subset \mathbb{P}(V)$ are the general upper bounds on complexity in 2.4 realized? In Section 4 we give an example of a single variety (in each codimension and embedding dimension) for which *all* the complexity achieves these upper limits.

3. SYSTEMS OF PARAMETERS FOR EQUIDIMENSIONAL MONOMIAL RINGS

Consider a polynomial ring $K[X_1, \dots, X_N]$ over a field K, and let I be any ideal generated by monomials $X_1^{e_1} X_2^{e_2} \cdots X_N^{e_N}$ in the variables X_i. The quotient ring

$$R = \frac{K[X_1, \dots, X_N]}{I}$$

is called a monomial ring. The purpose of this section is to combinatorially characterize all linear systems of parameters for equidimensional monomial rings.

3.1. LEMMA. *Let R be any graded (or local) Noetherian ring of dimension d and let $Y_1, \dots, Y_d$ be a collection of d elements in R. The following are equivalent.*

(1) *The elements $Y_1, \dots, Y_d$ are a system of parameters for R.*

(2) *The images of the elements $y_1, \dots, y_d$ are a system of parameters for the ring $R_{\mathrm{red}} = R/N$ obtained as the quotient by the ideal N of nilpotent elements in R.*

(3) *The images of the Y_i generate a nilpotent ideal modulo each minimal prime of R.*

If, in addition, dim $R/P = d$ *for all minimal primes P of R, then the above are equivalent to*:

(4) *The images of the elements $Y_1, \dots, Y_d$ are a system of parameters for every domain R/P, where P ranges through the minimal primes P of R.*

The proof is an easy exercise.

Geometrically, the lemma implies that the Noether complexity of a scheme in projective space is the same as the Noether complexity of the associated reduced subscheme, the variety obtained as the union of its irreducible components.

This has a particularly nice application to monomial rings. If I is a monomial ideal, then it has a primary decomposition

$$I = \bigcap_{i=1}^{r} \left(X_1^{a_{i1}}, \dots, X_N^{a_{iN}} \right),$$

where we omit the generator $X_i^{a_{ij}}$ when $a_{ij} = 0$. (There are algorithms for computing such a primary decomposition [3, 9, 15].) In particular, the minimal primes of a monomial ideal are generated by subsets of the variables. This leads to the following proposition.

3.2. PROPOSITION. *Let $I \subset k[X_1, \dots, X_N]$ be an equidimensional monomial ideal and let $P_i = (X_{i_1}, \dots, X_{i_{N-d}})$ for $i = 1, \dots, r$ be an enumeration of the minimal primes of I. Then a collection of linear forms $\{Y_i = \sum_{j=1}^{N} \lambda_{ij} X_j\}$ defines a system of parameters for the monomial ring $k[X_1, \dots, X_N]/I$ if and only if the matrix*

$$A = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1N} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2N} \\ \vdots & \vdots & \cdots & \vdots \\ \lambda_{d1} & \lambda_{d2} & \cdots & \lambda_{dN} \end{pmatrix}$$

satisfies the following rank condition: each of the r $d \times d$ subdeterminants formed by deleting the columns $\{i_1, \dots, i_{N-d}\}$ indexed by the minimal primes of I is nonzero.

Proof. By Lemma 3.1, the set $\{Y_i\}$ is a system of parameters for R if and only if its image modulo each P_i is a system of parameters for

$$\begin{aligned} R/P_i &\cong K[X_1, \dots, X_N]/\left(X_{i_1}, \dots, X_{i_{N-d}}\right) \\ &\cong K\left[X_1, \dots, \widehat{X_{i_1}}, \dots, \widehat{X_{i_{N-d}}}, \dots, X_N\right]. \end{aligned}$$

The latter ring is a polynomial ring in the d variables that are not generators for P_i.

It is easy to check that a set of d linear forms in a polynomial ring in d variables form a system of parameters for R if and only if they are linearly independent. Indeed, $\overline{Y}_d$ is dependent on the other $\overline{Y}_i$, where $\overline{Y}_i$ denotes reduction modulo $(X_{d+1}, \dots, X_N)$, if and only if the ideal generated by $(\overline{Y}_1, \dots, \overline{Y}_d)$ is the same as the ideal generated by $(\overline{Y}_1, \dots, \overline{Y}_{d-1})$. This holds if and only if $K[X_1, \ldots, X_d]/(\bar{Y}_1, \ldots, \bar{Y}_d) = K[X_1, \ldots, X_d]/(\bar{Y}_1, \ldots, \bar{Y}_{d-1})$ is not zero dimensional, that is, if and only if $Y_1, \ldots, Y_d$ is not a system of parameters for R.

This condition translates directly into the rank condition of Proposition 3.2. ∎

3.1

Proposition 3.2 allows one to easily check whether any set of linear forms is a system of parameters for a given equidimensional monomial ring, at least in theory. This does not mean that we have an explicit formula for Noether complexity, though it is clearly possible to describe, in terms of combinatorial data, how much overlapping occurs between the minimal primes.

We also note from Proposition 3.2 that the Plücker cobasis complexity for an equidimensional monomial ring is at least as large as the number of its minimal primes. This inequality relationship might be strict. For instance the Plücker complexity of the union of the coordinate spaces spanned by X_1, X_2 and X_3, X_4 is larger than 2.

3.2. *The Chow Form*

For any equidimensional projective variety $X \subset \mathbb{P}(V)$ of dimension $d+1$, the codimension d planes in $\mathbb{P}(V)$ that intersect X form a hypersurface in the Grassmannian $\mathbb{G}\mathrm{r}^d(V)$ of all codimension d planes in $\mathbb{P}(V)$.

We denote the Plücker (cobasis) coordinates for $\mathbb{G}\mathrm{r}^d(V)$ by $[j_1 j_2 \cdots j_d]$. This means that given a codimension d plane W, presented as the kernel of a $d \times N$ cobasis matrix A, the symbol $[j_1 j_2 \cdots j_d]$ denotes the determinant of the $d \times d$ matrix formed by the columns $j_1 < j_2 < \cdots < j_d$. The Plücker coordinates of a point in $\mathbb{G}\mathrm{r}^d(V)$ are well defined up to constant nonzero multiple.

The hypersurface in $\mathbb{G}\mathrm{r}^d(V)$ of planes intersecting X nontrivially is the vanishing set of a single polynomial F_X in Plücker coordinates: F_X is a *Chow form* of X. Technically speaking, this makes sense as stated only up to the radical, but for reduced X, the Chow form has no repeated factors. The degree of polynomial F_X is the degree of the variety X in $\mathbb{P}(V)$. The form $F_X \in \mathbb{P}(\mathrm{Sym}^r(\wedge^d V^*))$ is the point corresponding to X on the Chow variety parametrizing all degree r and dimension $d-1$ subvarieties of $\mathbb{P}(V)$. See [8] or [2] for more on Chow forms and Chow varieties. For the definition of Chow polytopes and their relation to Chow forms, see [10].

Despite—or because of—their importance, Chow forms are quite complicated and difficult to compute. Though the Chow form can be computed singly in exponential time [1], this is unfeasible for most interesting varieties. From a theoretical point of view, we prefer to have some general results for nice families of varieties, such as determinantal varieties. This has been accomplished for determinantal varieties of maximal minors and of 2-minors.

3.3

The Noether complexity can be read off the Chow form, at least in theory. The Noether complexity of $X \subset \mathbb{P}(V)$ is the least number of variables c_{ij} appearing in any initial monomial of F_X as we range over all term orders in $K[c_{ij}]$ [5, 2.7]. Here the c_{ij} are the indeterminate coefficients of the $\operatorname{codim}(X) \times \dim V$ matrix whose maximal minors are the Plücker coordinates.

Using Proposition 3.2, the Chow form of the varieties under consideration in this section is easy to write down.

3.3. COROLLARY. *Let $Z = V(I) = \subset \mathbb{P}^{N-1}$ be defined by a monomial ideal I of pure height $N - d$, i.e., Z is a union of coordinate hyperplanes all of the same dimension. Then the Chow form for Z is*

$$\prod_{1 \leq j_1 < \cdots < j_d \leq N} [j_1 \cdots j_d],$$

where the product is taken over all indices $j_1, \ldots, j_d$, such that the sets of elements $\{X_1, \ldots, X_N\} - \{X_{j_1}, \ldots, X_{j_d}\}$ generate the minimal primes of I (equivalently, define an irreducible component of Z). By definition, a codimension d plane in $\mathbb{P}^{N-1}$ intersects Z nontrivially if and only if it Plücker (cobasis) coordinates satisfy this equation.

From this expression, we confirm that equidimensional varieties that are unions of coordinate planes have degree equal to the number of their irreducible components.

4. A MAXIMALLY COMPLEX VARIETY

In this section, we study monomial rings closely related to the determinantal varieties. These turn out to be an intersecting class of varieties because they are maximally complex, for every sense of the Noether complexity discussed in Section 2. We will make use of these results in the next sections.

4.1. *Notation*

Let $S = K[X_1, X_2, \ldots, X_N]$ and let I be the ideal generated by *all* the degree $t + 1$ square-free monomials in the variables $X_1, \ldots, X_N$. Here, N is assumed larger than t.

Let Z_t be the subvariety of $\mathbb{P}^{N-1} = \mathbb{P}(V)$ defined by I and let $R = R(t, N) = S/I$ be its homogeneous coordinate ring. As before, we think of a point in $\mathbb{P}^{N-1}$ as an element in the space V of $N \times 1$ column vectors; the coordinate functional X_i plucks out the ith row; cf. Section 2.1. The variety $Z_t \subset \mathbb{P}(V)$ has a combinatorial description as the set of all column vectors in V of complexity at most t; see the proof of Proposition 4.2.

4.2. *Primary Decomposition of I*

Let

$$J_{\{i_1, i_2, \ldots, i_t\}} = \left(X_1, X_2, \ldots, \hat{X}_{i_1}, \ldots, \hat{X}_{i_t}, \ldots, X_N \right)$$

be the ideal of S generated by all the variables X_i except $X_{i_1}, X_{i_2}, \ldots, X_{i_t}$. One easily checks that the minimal primes of I are exactly the prime ideals $J_{\{i_1, i_2, \ldots, i_t\}}$ as $\{i_1, i_2, \ldots, i_t\}$ ranges over all possible t-tuples of integers $1 \leq i_1 < i_2 < \cdots < i_t \leq N$. Because I is radical, the intersection of these ideals is a primary decomposition for I. Also, because the height of each such $J_{\{i_1, i_2, \ldots, i_t\}}$ is $N - t$, we see that $R(t, N) = S/I$ is equidimensional of dimension t.

In other words, the variety $Z_t \subset \mathbb{P}(V)$ is the union of *all* the $t - 1$-dimensional coordinate planes in $\mathbb{P}(V)$. This variety is of pure dimension $t - 1$.

4.1. PROPOSITION. *The Noether cobasis complexity of R is $Nt - (t - 1)t$. Furthermore, every linear space of $\mathbb{P}(V)$ of maximal dimension disjoint from Z_t has cobasis complexity exactly $tN - (t - 1)t$.*

Proof. Let $Y = \lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_N X_N$ be any linear form in S. If the complexity of Y is less than $N - t + 1$, then Y is contained in some ideal $J_{\{i_1, i_2, \ldots, i_t\}}$. (For instance, one may choose any set of indices $\{i_1, i_2, \ldots, i_t\}$ such that the corresponding coefficients $\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_t}$ are all zero: the bound on the complexity of Y ensures the existence of at least t distinct such indices i_j.)

Therefore, given a linear system of parameters $Y_j = \sum_{i=1}^{N} \lambda_{ji} X_N$ for S/I, each of the t parameters Y_j must have complexity at least $N - t + 1$. This gives the immediate lower bound on the Noether complexity of I, namely,

$$t(N - t + 1).$$

On the other hand, this lower bound is also an upper bound, by 2.4. In fact, any minimally complex system of parameters can be assumed to have the form

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \cdots & \lambda_{1N-t+1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_{22} & \lambda_{23} & \cdots & \cdots & \lambda_{2N-t+2} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_{33} & \lambda_{34} & \cdots & \cdots & \lambda_{3N-t+3} & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & & \lambda_{t-2t-2} & \cdots & \cdots & \lambda_{t-2N-2} & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \lambda_{t-1t-1} & \cdots & \cdots & \lambda_{t-1N-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \lambda_{tt} & \cdots & \cdots & \lambda_{tN} \end{pmatrix}, \tag{4.1.1}$$

after suitable row operations or permutations of rows and columns. This matrix exhibits a system of parameters for S/I which has the desired minimal complexity, assuming the λ_{ij} are suitably generic. Indeed, from Proposition 3.2, we know that we have a parameter matrix if and only if *all* the t minors of this matrix are nonzero.

An alternative system of parameters which generates the same ideal in S is obtained by multiplying on the left by a $t \times t$ invertible matrix so as to get

$$\left(\mathrm{Id}(t \times t), \qquad B(t \times (N-t))\right),$$

where $\mathrm{Id}(t \times t)$ is a $t \times t$ identity matrix and $B(t \times (N-t))$ is a some matrix of size $t \times (N-t)$. The matrix $B(t \times N - t)$ is completely dense: all entries and all t minors are nonzero.

This shows that every liner space of codimension t disjoint from X has complexity $tN - t(t-1)$, which is maximal possible by 2.4. The parameter matrix is the same as a cobasis representative for a linear space of codimension t in $\mathbb{P}^{N-1}$ disjoint from Z_t. Thus the cobasis complexity of every such linear space is $Nt - (t-1)t$, and Z_t has Noether complexity $Nt - (t-1)t$, the maximum allowable by 2.4. ∎

Finally, we treat the basis complexity of $Z_t \subset \mathbb{P}(V)$.

4.2. PROPOSITION. *Let $Z_t \subset \mathbb{P}(V)$ be the variety as defined in Section* 4.1. *Then a codimension t plane $\mathbb{P}(W)$ in $\mathbb{P}(V)$ is disjoint from Z_t if and only* *if some (equivalently, every) $N \times (N-t)$ matrix whose columns span W has all $N-t$ minors nonzero. Thus every linear space of codimension t disjoint form Z_t has complexity $(t+1)(N-t)$.*

Proof. The variety $Z_t \subset \mathbb{P}(V)$ has a nice combinatorial description: it is the set of all column vectors in V of complexity at most t. Indeed, if

$$v = \begin{bmatrix} \vdots \\ \lambda_{i_k} \\ \vdots \end{bmatrix}$$

has complexity greater than t, then some set of $t+1$ coefficients λ_i are all nonzero. This means that some $X_{i_1} \cdots X_{i_{t+1}}$ do not vanish at v, so v is not in Z_t. Conversely, if v has complexity at most t, then every set of $t+1$ coefficients λ_i contains some $\lambda_i = 0$ and every $X_{i_1} \cdots X_{i_{t+1}}$ vanishes on v.

Let B be an arbitrary $N \times N - t$ matrix and let $W \subset V$ be the subspace spanned by its columns. We claim that every element in W has complexity at least $t+1$ if and only if all $N-t$ minors of B are nonzero.

To see this, suppose first that some $(N-t)$ minor vanishes, say the one obtained from the first $(N-t)$ rows of B. Then B is column-equivalent to a matrix of the form

$$\begin{pmatrix} \mathrm{Id}(r \times r) & \mathbf{0}(r \times N - t - r) \\ B_1(N - t - r \times r) & \mathbf{0}(N - t - r \times N - t - r) \\ B_2(t \times r) & B_3(t \times N - t - r) \end{pmatrix},$$

where the B_i are arbitrary matrices of the indicated sizes, $\mathrm{Id}(r \times r)$ is an $r \times r$ identity matrix, and the $\mathbf{0}$s are zero matrices of the indicated sizes. The space W therefore contains the vector appearing as the last column of this matrix and it has complexity at most t.

Conversely, if W contains a vector of complexity at most t, then we choose a basis for W containing this vector: the corresponding basis matrix B' has a column with at least $N-t$ zeros. Thus some $N-t$ minor of B' is zero, and because B and B' are column-equivalent, we conclude that B has the same property.

As in the proof of Proposition 4.1, the complexity of any $N \times N - t$ matrix B in which all maximal minors are nonzero is at least $(t+1)(N-t)$. Because this number is also an obvious upper bound on the basis complexity of W, we conclude that the basis complexity of every linear space of maximal dimension that is disjoint from Z_t is exactly $(t+1)(N-1)$. ∎

4.3. COROLLARY. *The variety* $Z_t \subset \mathbb{P}(V)$ *is maximally complex in every sense*:

(1) *The Noether (cobasis) complexity is* $Nt - (t-1)t$.

(2) *The Noether basis complexity is* $(t+1)(N-t)$.

(3) *The Noether Plücker complexity, both basis and cobasis, is* $\binom{N}{t}$.

All of these are maximally possible for an arbitrary linear space in $\mathbb{P}(V)$ *of codimension* t.

5. SPARSE SYSTEMS OF PARAMETERS FOR DETERMINANTAL VARIETIES

The purpose of this section is to describe a sparse and convenient system of parameters for determinantal varieties. It is also a system of parameters for the variety defined by the monomial ideal of leading terms in the standard diagonal term order. We prove that this system of parameters is maximally sparse among those that admit a convenient symmetry of being "partitioned along diagonals." In Section 7, we will show that in certain cases it is the sparsest possible among *all* system of parameters, but we also give an example to show that it is not the sparsest possible in general. On the other hand, in Section 6, we prove that the linear subspace of projective space it defines is the sparsest possible in the basis sense.

5.1. *Notation*

Let V be the vector space of $m \times n$ matrices with coefficients in K and suppose $m \leq n$. Fix the standard basis $\{e_{ij}\}$ for V of matrices: e_{ij} has zeros in each position except the ijth position, where there is a 1. Let X_{ij} be the dual basis of coordinates.

The determinantal variety $X_t = X_t(m, n) \subset \mathbb{A}(V)$ [or $\mathbb{P}(V)$] is the subvariety of matrices of rank less than or equal to t. The variety X_t is defined by the vanishing of the size $t+1$ minors of the $m \times n$ matrix $\mathbf{X}_{ij}$. The coordinate ring $R_t(m \times n)$ for X_t is the quotient of the polynomial ring $K[X_{ij}]$ by the determinantal ideal $I_{t+1}(m \times n)$ generated by the size $t+1$ subdeterminants of $\mathbf{X}_{ij}$.

It is easy (see, e.g., [8, p. 151]) to check that the codimension of $X_t \subset \mathbb{P}(V)$ is $(m-t)(n-t)$. We set $d = mn - (m-t)(n-t) = nt + mt - t^2$; this is the dimension of the affine cone over X_t in $\mathbb{A}(V)$.

5.1.1

The monomials of the polynomial ring $k[X_{ij}]$ can be ordered by the "diagonal term order": the lexicographic ordering given by the ordering

$$X_{1n} < X_{1(n-1)} < \cdots < X_{11} < X_{2n} < X_{2(n-1)} < \cdots < X_{m2} < X_{m1}$$

of the variables. In this case, the $t+1$ minors of (X_{ij}) are a Gröbner basis for the ideal $I_{t+1}(m,n)$ [12].

The Noether complexity of I is bounded above by the Noether complexity of the initial ideal of I in general, because a system of parameters for $S/\text{init}(I)$ can be suitably lifted to a system of parameters for S/I of the same complexity [5, 2.8]. Interestingly, determinantal varieties seem to be as complex as they can possibly be in this respect, and we believe that the Noether complexity of a determinantal ideal may equal the Noether complexity of its initial ideal, though we have proved this only for $t = 1, 2, m-1$.

We now describe a family of linear forms, $\mathcal{S}$, that are systems of parameters for $(K[\mathbf{X}_{ij}])/I_{t+1}$, and also for $(K[\mathbf{X}_{ij}])/\text{init}(I_{t+1})$.

5.1.2. *A Partition*

The set $\mathcal{S}$ of parameters will be "partitioned along diagonals" in the following sense. The linear forms Y_i making up the system of parameters will be partitioned into sets $\mathcal{S}_k$ of forms made up of variables lying only on the kth diagonal D_k of $\mathbf{X}_{ij}$. More precisely, the variables X_{ij} are partitioned into sets

$$D_1 = \{X_{11}\}, \qquad D_2 = \{X_{12}, X_{21}\}, \qquad D_3 = \{X_{13}, X_{22}, X_{31}\}, \ldots,$$

where

$$D_k = \left\{X_{ij} \text{ such that } i + j = k + 1\right\}$$

consists of the elements lying on the kth (anti-)diagonal of the matrix $\mathbf{X}_{ij}$. The set $\mathcal{S}_k$ will consist of linear forms in the variables from D_k.

5.1.3. *The System of Parameters*

Let $\mathcal{S}$ be the following set of linear forms in $K[\mathbf{X}_{ij}]$:

$$\mathcal{S} = \bigcup_{k=1}^{m+n-1} \mathcal{S}_k,$$

where each $\mathcal{S}_k$ consists of linear combinations of the variables in the set D_k, described as follows.

We first describe $\mathcal{S}_k$ for $k \leq t$ and $k \geq n - m - t$, corresponding to the "top left" and "bottom right" corners of $\mathbf{X}_{ij}$. Fix any $|D_k| \times |D_k|$ matrix with nonzero determinant, where $|D_k|$ denotes the cardinality of D_k. Then $\mathcal{S}_k$ is the set of linear forms in the elements of D_k whose coefficients are the rows of this matrix. Each such $\mathcal{S}_k$ has the same cardinality as D_k. For example, a maximally sparse way to do this is to choose the $|D_k| \times |D_k|$ identity matrix; in this case, each $\mathcal{S}_k = D_k$. The elements in these $\mathcal{S}_k$s contribute a total of $t(t+1)$ elements to the set $\mathcal{S}$.

Each of the remaining sets $\mathcal{S}_k$ is defined as follows: take any $t \times |D_k|$ matrix with the property that no t minor vanishes. The elements of $\mathcal{S}_k$ are the linear forms $Y_1, \dots, Y_t$ whose coefficients are the rows of this matrix. Thus for each k, $t < k < n + m - t$, the set $\mathcal{S}_k$ has cardinality t and consists of linear forms in the variables of D_k. For example, the maximally sparse ways of doing this are discussed in Proposition 4.1 [see matrix (4.1.1)].

5.2. THEOREM. *The elements of the set* $\mathcal{S} = \bigcup_{k=1}^{n+m-1} \mathcal{S}_k$ *form a system of parameters for the determinantal ring* $R = (K[\mathbf{X}_{ij}])/I_{t+1}$. *Every ideal generated by a system of parameters that can be partitioned along the diagonals of* $\mathbf{X}_{ij}$ *is generated by a system of parameters of this form.*

Proof. The dimension of R is $d = nt + mt - t^2$ and this is also the cardinality of $\mathcal{S}$. Thus to check that the elements $\{Y_1, \dots, Y_d\}$ are a system of parameters, we need only verify that every variable X_{ij} of $K[\mathbf{X}_{ij}]$ becomes nilpotent modulo $I_{t+1} + (\mathcal{S}) = I_{t+1} + (Y_1, \dots, Y_d)$.

We accomplish this "diagonal by diagonal," using induction on k, the index for the diagonal sets.

For $k \leq t$ and $k \geq n + m - t$, this is easy. The ideal generated by $\mathcal{S}_k$ is the same as the ideal generated by D_k, since we can left-multiply the matrix whose rows define $\mathcal{S}_k$ by its inverse without affecting the ideal. So the T variables X_{ij} in D_k are in the set $\mathcal{S}$, whence they are certainly nilpotent modulo $I_{t+1} + (\mathcal{S})$.

Assume inductively that all the elements of the diagonal sets $D_1, D_2, \dots, D_{k-1}$ are nilpotent modulo $I_{t+1} + (\mathcal{S})$, where $k - 1 \geq t$. To show the same for D_k, it is enough to show that the elements of D_k are nilpotent modulo $I_{t+1} + (Y_1, \dots, Y_d) + (D_1) + (D_2) + \cdots + (D_{k-1})$. Note that this ideal contains all the square-free monomials of degree $t + 1$ in the variables from the set D_k.

By our choice of the elements $Y_1, Y_2, \dots Y_t$ of $\mathcal{S}_k$ (re-indexing if necessary), we know by Theorem 4.1 that these elements constitute a system of parameters for the ring

$$R_k = \frac{K\left[X_{1(k+1)}, X_{2k}, \ldots, X_{(k+1)1}\right]}{(\text{all square-free monomials of deg } t+1)}.$$

By definition, then, each coordinate X_{ij} of R_k is nilpotent modulo the ideal generated by the t linear forms $Y_1, Y_2, \ldots, Y_t$. Thus each element of the diagonal set $D_k \subset K[\mathbf{X}_{ij}]$ is nilpotent modulo the ideal J generated by all the square-free monomials of degree $t+1$ in the variables in D_k plus the ideal $\mathcal{S}_k$. Because

$$J + \mathcal{S}_k \subset I_{t+1} + (Y_1, \ldots, Y_d) + (D_1) + (D_2) + \cdots + (D_{k-1}),$$

we conclude that each $X_{ij} \in D_k$ is nilpotent modulo the desired ideal. It follows that $Y_1, \ldots, Y_d$ are a system of parameters for the determinantal ring $R = (K[\mathbf{X}_{ij}])/I_{t+1}$.

Finally, we show that every system of parameters partitioned along the diagonal has this form. Suppose that $\mathcal{S} = \{Y_1, \ldots, Y_d\}$ can be partitioned into sets $\mathcal{S}_k$ such that the elements of $\mathcal{S}_k$ are linear combinations of the elements D_k of the kth diagonal. For $t < k < m + n - t$, the natural surjection

$$R/(\mathcal{S}) \twoheadrightarrow R/((\mathcal{S}) + (D_1) + \cdots (\text{omit } D_k) \cdots + (D_{m+n-1})) \cong R_k/(\mathcal{S}_k)$$

shows that $R_k/(\mathcal{S}_k)$ is zero dimensional; therefore, because R_k has dimension t, the cardinality of each D_k is at least t. Similarly, the remaining $\mathcal{S}_t$ must have cardinality at least $|D_k|$. A dimension count now shows that the cardinality of each $\mathcal{S}_k$ is *exactly* t for $t < k < m + n - t$, and exactly $\mathcal{S}_k$ for the remaining choices of k. In both cases, the elements of $\mathcal{S}_k$ therefore form a system of parameters for R_k (which is interpreted as simply a polynomial ring in the variables of D_k when $t > |D_k|$), and the proof is complete. ∎

The above procedure gives a natural way to choose sparse systems of parameters for the determinantal ideal $I_{t+1}(m \times n)$. The added symmetry of diagonal partitioning is helpful in computations. The next proposition shows that it is the sparsest possible among all systems of parameters that can be partitioned in this way.

5.3. COROLLARY. *The cobasis complexity of the linear space defined by the linear forms* $\mathcal{S}$ *described in Section* 5.1.3 *is*

$$tmn - (t-1)d.$$

This is the sparsest possible among systems of parameters that can be partitioned along diagonals. Its Plücker (cobasis) complexity is

$$\binom{m}{t}^{n-m+1}\binom{t+1}{t}^2\binom{t+2}{t}^2\cdots\binom{m-1}{t}^2.$$

Proof. Because the elements of $\mathcal{S}_k$ form a system of parameters for $R_k \cong R(t, |D_k|)$ (notation of Sections 5.1 and 5.1.3), we know from Proposition 4.1 that the minimal possible complexity of each $\mathcal{S}_k$ is $t(|D_k| - t + 1)$ if $m + n - 1 - t \geq |D_k| \geq t$ and is $|D_k|$ otherwise. Furthermore, given any such $\mathcal{S}_k$, the ideal generated by its elements has a generating set of exactly these minimal complexities. Thus the parameter matrix can be replaced by a row-equivalent matrix of complexity

$$t(t+1) + \sum_{k=t+1}^{n+m-t-1} t\big(|D_k| - t + 1\big) = tmn - (t-1)d.$$

This is the complexity of every linear space cut out by a system of parameters partitioned along the diagonals.

Let $\mathcal{A}$ be a parameter matrix of complexity $tmn - (t-1)d$. If $\mathcal{A}$ is partitioned along diagonals, then $\mathcal{A}$ has block diagonal form, in which appears a single $(t+1)t \times t(t+1)$ full rank identity matrix (grouping together all the "corner" $\mathcal{S}_k$s) and blocks of size $t \times |D_k|$, for $k = t + 1, \ldots, m + n - t$, each block corresponding to the remaining diagonals of $\mathbf{X}_{ij}$. Each of these blocks has minimal complexity among all row-equivalent blocks by Proposition 4.1, and so $\mathcal{A}$ has minimal complexity among all row-equivalent matrices. Thus, the cobasis complexity of the linear space defined by the linear forms $\mathcal{S}$ is exactly $tmn - (t-1)d$.

Finally, we compute the Plücker complexity. A $d \times d$ submatrix $\mathcal{B}$ of $\mathcal{A}$ has full rank if and only if it consists of the $t(t+1)$ full rank block together with exactly t columns from each $t \times |D_k|$ block.

Indeed, given such a matrix $\mathcal{B}$, it must have full rank because every t minor of the $t \times |D_k|$ blocks is of full rank. Conversely, if $\mathcal{B}$ is missing a column from the size $t(t+1)$ identity matrix, then $\mathcal{B}$ has a row of zeros; likewise, if $\mathcal{B}$ has at least $t + 1$ columns from a $t \times |D_k|$ block, then these $t + 1$ columns are linearly dependent, so $\mathcal{B}$ cannot have full rank.

The number of choices of a full rank t minor from the kth diagonal is $\binom{|D_k|}{t}$, and since these choices are independent of each other, the total number of nonzero maximal minors of the parameter matrix is

$$\binom{m}{t}^{n-m+1}\binom{t+1}{t}^2\binom{t+2}{t}^2\cdots\binom{m-1}{t}^2. \qquad \blacksquare$$

The method explained here for coming up with a sparse system of parameters for determinantal rings also works for similar rings, e.g., ladder determinantal rings.

Finally, we point out that the linear forms $\mathcal{S}$ are also a linear system of parameters for the associated variety $\mathbb{V}(\text{init}(I))$ formed from the initial ideal of $I_{t+1}(m \times n)$ with respect to the diagonal term order described in Section 5.1.1.

Recall that the $t+1$ minors of $\mathbf{X}_{ij}$ are a Gröbner basis with respect to the diagonal term order. This means that the ideal $\text{init}(I)$ is generated by the degree $t+1$ square-free monomials $X_{i_1 j_1} X_{i_2 j_2} \cdots X_{i_{t+1} j_{t+1}}$, where the sequence $i_1, \dots, i_{t+1}$ is strictly decreasing and the sequence $j_1, j_2, \dots, j_{t+1}$ is strictly increasing.

5.4. PROPOSITION. *The linear forms $\mathcal{S}$ described in Section* 5.1.3 *are a system of parameters for $(K[X_{ij}])/\text{init } I_{t+1}$ of minimal complexity among those that can be partitioned along diagonals. In particular, the Noether complexity of this ring is at most $mnt - (t-1)d$.*

Proof. The fact that the elements of $\mathcal{S}$ are a system of parameters for $(K[X_{ij}])/\text{init } I_{t+1}$ is easy to see. Note that $(K[X_{ij}])/\text{init } I_{t+1}$ is a homomorphic image of the ring

$$S = \bigotimes_{k=1}^{m+n-1} R_k,$$

where

$$R_k \cong \frac{K[\text{variables in } D_k]}{(\text{all square-free degree } t+1 \text{ monomials})}$$

is the ring studied in Section 4 (or for $k \le t$, it is the polynomial ring in k variables). Here, k indexes the diagonals of $\mathbf{X}_{ij}$ and ranges from 1 to $m+n-1$. Each set $\mathcal{S}_k$ is defined to be a system of parameters for R_k, so that obviously their union is a system of parameters for the tensor product.

By computing the dimension of S we see that it is the same as the dimension of $(K[X_{ij}])/\text{init } I_{t+1}$, so their images under the natural surjection form a system of parameters for $(K[X_{ij}])/\text{init } I_{t+1}$.

Each $\mathcal{S}_k$ is a minimally complex system of parameters for R_k, as proved in Proposition 4.1. However, now any system of parameters for $(K[X_{ij}])/\text{init } I_{t+1}$ that is partitioned along diagonals must lift to a system of parameters for S. Indeed, as in the proof of Proposition 5.2, the parameters involving elements of the kth diagonal must be a system of parameters for the ring obtained by killing all variables not on this diagonal. So again by the results of Section 4, we conclude that $\mathcal{S}$ is a sparsest possible system of parameters among systems of parameters partitioned along the diagonals. ∎

It is tempting to believe that Noether complexity is additive in tensor products. This is false! See Example 7.12.

6. BASIS COMPLEXITY FOR DETERMINANTAL VARIETIES

The purpose of this section is to prove the following formula for the basis complexity of determinantal varieties.

6.1. THEOREM. *Let $X_t(m \times n)$ be the variety of $m \times n$ matrices of rank at most t. The Noether basis complexity of $X_t(m, n)$ is $(m - t)(n - t)(t + 1)$. The Noether basis complexity of the variety defined by the initial ideal of $I_{t+1}(m \times n)$ is at most $(m - t)(n - t)(t + 1)$.*

An explicit (and sparsest possible) basis for a linear space (of maximal dimension) disjoint from X_t is

$$A_{k,r} = \begin{pmatrix} 0 & 0 & \cdots & & & \cdots & 0 & 0 \\ \vdots & \vdots & & & & & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \lambda_{kr}^{(t)} & & 0 \\ 0 & \cdots & & & \lambda_{kr}^{(t-1)} & 0 & \cdots & 0 \\ \vdots & & & \ddots & & & & 0 \\ 0 & & \lambda_{kr}^{(0)} & & \cdots & 0 & & 0 \\ \vdots & & & \cdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}.$$

For each pair (k, r) with $t \le k < m + n - t$ and $0 \le r < |D_k| - t$, let $\lambda_{k,r}^{(0)}, \dots, \lambda_{k,r}^{(t)}$ be suitably generic members of K and define the matrix $A_{k,r}$ to be the $m \times n$ matrix that is zero off the kth antidiagonal D_k, whose entries $X_{k-r,r+1}, \dots, X_{k-r-t,r+t+1}$ on D_k are $\lambda_{k,r}^{(0)}, \dots, \lambda_{k,r}^{(t)}$, respectively, and whose other entries on D_k are zero. That is, $A_{k,r}$ is the $m \times n$ matrix with zeros everywhere except on a single antidiagonal, on which exactly $t + 1$ contiguous nonzero entries appear. The precise meaning of "suitably generic" is: for each set of basis elements $A_{k,r}$ involving nonzero elements on the same diagonal D_k, the $(|D_k| - t) \times |D_k|$ matrix formed from them must have all its maximal minors nonzero. The $m \times n$ matrices $A_{k,r}$ are a collection of $(n-t)(m-t)$ matrices of rank $t+1$. We will check below that they span a linear space of maximal dimension disjoint from X_t.

The linear space spanned by the matrices above is easily seen to be defined by a system of parameters $\mathcal{S}$ partitioned along diagonals, as described in Section 5.1.3. So $\mathcal{S}$ gives a cobasis representation for a minimally complex linear space (with respect to basis representation).

Proof of Theorem 6.1. The variety $X_t(m,n) \subset \mathbb{A}(\mathbb{C}^{m\times n})$ consists of exactly the matrices of rank less than or equal to t. Thus, if $W \subset V \cong \mathbb{C}^{m\times n}$ intersects X_t trivially, then *every* nonzero matrix in W has rank at least $t+1$. Because X_t has codimension $(n-t)(m-t)$, there is some such W of dimension $(n-t)(m-t)$. Any basis for W consists of $(n-t)(m-t)$ matrices of rank at least $t+1$. Thus an obvious *lower bound* on the basis complexity of W is $(t+1)(n-t)(m-t)$.

This lower bound is also an upper bound on the complexity, because the linear space W given by the vanishing of the elements of $\mathcal{S}$ described in Section 5.1.3 decomposes as $W \cong \bigoplus_{k=1}^{m+n-1} W_k$, where the W_k are indexed by the diagonals of $\mathbf{X}_{ij}$. Each W_k is a subspace of the vector space V_k spanned by the vectors e_{ij} (standard matrix notation), where the index ij is on the kth diagonal.

Because the elements of W all have rank greater than t, the subspace W_k of *diagonal* matrices contains only the linear subspace of vectors in V_k of complexity greater than t. In fact, computing its dimension, we see that it is a linear subspace of V_k of maximal dimension containing no element (except zero) of complexity less than $t+1$; c.f. Section 4. We have seen that such a space is a linear subspace of maximal dimension disjoint from the varieties $Z_t \subset \mathbb{P}(V_k)$. From Proposition 4.2, we know that W_k has complexity exactly $(t+1)(|D_k| - t)$ for $k > t$. For $k \le t$, $W_k = 0$, since no matrix in this set has rank greater than t. Summing over k, we arrive at the complexity $(t+1)(n-t)(m-t)$ for W. This shows that the Noether basis complexity for determinantal variety $X_t(m \times n)$ is precisely $(t+1)(n-t)(m-t)$.

The matrices $A_{k,r}$ described following Theorem 6.1 for each fixed k are a basis for each W_k, by Theorem 4.2, assuming the correct genericity assumption on the entries as described. We can conclude that all the $A_{k,r}$ for all k are a basis for W. This basis has complexity $(t+1)(m-t)(n-t)$, and the proof is complete.

By the construction, it is clear that the space W we have described is also disjoint from the variety defined by the initial ideal of $I_{t+1}(m \times n)$.

Thus the Noether basis complexity of

$$\frac{K[X_{ij}]}{\operatorname{init}(I)}$$

is at most $(t+1)(m-t)(n-t)$. ∎

7. NOETHER COMPLEXITY OF DETERMINANTAL VARIETIES

In this section, we prove that the systems of parameters described in Section 5 are the sparsest possible in certain cases. We also provide Example 7.13 to show that, by breaking the diagonal symmetry in the systems of parameters, we can construct even sparser systems.

7.1. THEOREM. *For $t = 1, 2$, or $m - 1$, the Noether cobasis complexity of the determinantal variety $X_t(m, n)$ is*

$$tmn - (t-1)(tn + tm - t^2) = tmn - (t-1)d,$$

where d denotes the dimension of $(K[\mathbf{X}_{ij}])/I_{t+1}$. The same formula applies for the variety defined by the initial ideal of $I_{t+1}(m \times n)$.

We first focus on the proof of Theorem 7.1 in the cases $t = 1$ and $t = 2$. The following lemma will be useful.

7.2. LEMMA. *Let I be a homogeneous ideal of $K[X_1, \dots, X_N]$ that is contained in the ideal $J_{\{i_1, i_2, \dots, i_t\}}$ generated by all the variables X_i except $X_{i_1}, X_{i_2}, \dots, X_{i_t}$. Then any parameter matrix A for I has the property that the $d \times t$ submatrix of A determined by the columns indexed $i_1, i_2, \dots, i_t$ has rank t.*

Proof. Let $A = (\lambda_{ij})$ be any parameter matrix for I. The $d \times N$ matrix A represents the system of parameters $Y_i = \sum_{j=1}^{N} \lambda_{ij} X_i$ for $(K[X_1, X_2, \dots X_N])/I$. In particular, the ideal $I + (Y_1, Y_2, \dots, Y_d)$ of $K[X_1, X_2, \dots X_N]$ is height N. Thus the larger ideal, $J_{\{i_1, i_2, \dots, i_t\}} + (Y_1, Y_2, \dots, Y_d)$ also has height N. However, modulo this ideal, we are left with a ring isomorphic to

$$\frac{K[X_{i_1}, X_{i_2}, \dots, X_{i_t}]}{\left(\overline{Y}_1, \overline{Y}_2, \dots, \overline{Y}_d\right)},$$

where the overbar indicates reduction modulo the ideal generated by all the variables except $X_{i_1}, X_{i_2}, \dots, X_{i_t}$. Indeed, the coefficients of the $\overline{Y}_j$s are given by the $d \times t$ submatrix of A determined by the columns indexed $i_1, i_2, \ldots, i_t$. Of course, any (invertible) linear change of variables produces an isomorphic ring, so that after suitable linear coordinate change, this ring is seen to be isomorphic to the ring

$$\frac{K\left[X'_{i_1}, X'_{i_2}, \ldots, X'_{i_t}\right]}{\left(X'_{i_1}, X'_{i_2}, \ldots, X'_{i_s}\right)},$$

where s is the rank of this $d \times t$ matrix. Since this quotient ring is zero dimensional, we see that $s = t$. Therefore, the $d \times t$ submatrix of A determined by the columns indexed $i_1, i_2, \ldots, i_t$ has rank t. ∎

7.3. DEFINITION. Let A be a $d \times N$ matrix. We say that A has the tth maximal rank property (or just property M_t, for short) if, given any t columns of A, the $d \times t$ submatrix of A they determine has rank t.

If M_{t+1} holds for any matrix, then clearly M_t also holds.

7.4. LEMMA. *Every parameter matrix of the determinantal ring* $(K[X_{ij}])/(I_{t+1}(m \times n))$ *has the tth maximal minor property.*

Proof. Consider a graded ring $R = (K[X_1, \ldots, X_N])/I$ such that

$$I \subset J_{\{i_1, i_2, \ldots, i_t\}},$$

where $J_{\{i_1, \ldots, i_t\}}$ is the ideal generated by all the variables *except* the designated set $\{X_{i_1}, X_{i_2}, \ldots, X_{i_t}\}$ of t variables. Any parameter matrix must have full rank in the submatrix formed by the t columns indexed by $1 \leq i_1 < i_2 < \cdots < i_t \leq N$.

Now note that I_{t+1} is contained in each ideal $J_{\{i_1, \ldots, i_t\}}$ for every set of indices $1 \leq i_1 < i_2 < \cdots < i_t \leq N$. Indeed, the generators of I_{t+1} are sums of square-free monomials of degree $t + 1$, so the ideal generated by these monomials. This monomial ideal, and therefore the ideal $I_{t+1}(m \times n)$, is contained in the ideal $J_{\{i_1, \ldots, i_t\}}$, as every degree $t + 1$ square-free monomial is divisible by $t + 1$ distinct variables, and hence divisible by some variable not in the designated set.

This means that every parameter matrix for determinantal varieties satisfies the tth maximal rank property, M_t. ∎

We get some bounds on the Noether complexity of any rings whose parameter matrices satisfy condition M_t.

7.5. PROPOSITION. *Let* $R = (K[X_1, \ldots, X_N])/I$ *be a ring, all of whose parameter matrices satisfy condition* M_t. *If* $t = 1$, *then the Noether complexity of* I *is at least* N. *If* $t \geq 2$, *then the Noether complexity of* I *is at least* $2N - \dim R$.

Proof. First some general observations. Fix any $d \times N$ matrix A. For each integer h between 1 and d inclusive, let $l_h(A)$ (or simply l_h) be the number of columns of A in which at least h nonzero entries appear. With this notation, the complexity of A is exactly

$$l_1 + l_2 + \cdots + l_d.$$

In particular, if A is a parameter matrix of minimal complexity for the ring R, then denoting the Noether complexity of R by $\mathrm{NC}(I)$, we see that

$$\mathrm{NC}(I) \geq l_1(A) + l_2(A) + \cdots + l_k(A), \qquad (7.5.1)$$

for any $k \leq d$ equal to the dimension of R.

The first assertion of Proposition 7.5 is now easy to prove. If A has the first maximal rank property M_1, then every $d \times 1$ submatrix of A has rank 1. This means that each column of A has a nonzero entry, so that $l_1(A) = N$ for all parameter matrices A. In particular, we conclude that $\mathrm{NC}(I) \geq N$.

The second assertion requires more work. Fix a parameter matrix A for R which has minimal complexity. It suffices to prove that if A satisfies the second maximal rank property M_2, then its complexity is at least $2N - d$.

First an observation about A: if some row (say row i) has exactly one nonzero entry (say λ_{ij}), then the column in which this entry appears (column j) also has exactly one nonzero entry. Indeed, if the ith row is $(0, 0, \dots, 0, \lambda_{ij}, 0, \dots, 0)$, then one may perform elementary row operations of adding multiples of this row to the others without increasing the complexity of A. The assumption of minimality on the complexity of A therefore forces all the entries of column i to be zero except λ_{ij}.

Let h denote number of rows in which exactly one element appears. The argument of the preceding paragraph enables us to assume, after suitable reordering of the X_js and Y_is, that the minimally complex parameter matrix has the form

$$A = \begin{pmatrix} \mathrm{Id}(k \times k) & \mathbf{0}(k \times (N-k)) \\ \mathbf{0}((d-k) \times k) & B \end{pmatrix},$$

where $\mathrm{Id}(k \times k)$ denotes a $k \times k$ identity matrix, $\mathbf{0}(a \times b)$ denotes an $a \times b$ matrix of zeroes, and B is a $(d-k) \times (N-k)$ matrix such that every row contains at least two nonzero entries.

Now an observation about B: if two columns of B have exactly one nonzero entry, then they must occur in different rows. Indeed, consider the $d \times 2$ submatrix of A determined by the two columns indexed by j_1 and j_2 with $j_1, j_2 > k$. Because A has property M_2, this $d \times 2$ submatrix has rank 2, which means that if the columns each have exactly one nonzero entry (which will necessarily be in B), they must appear in different rows.

This observation about B ensures that each of the $d - k$ rows of B contains at most one entry λ_{ij} appearing in a column which has all entries zero except this λ_{ij}. Therefore, at least $(N - k) - (d - k) = (N - d)$ columns of B contain two or more elements. That is, $l_2(B)\,[= l_2(A)]$ is at least $N - d$.

We conclude, using (7.5.1), that

$$\mathrm{NC}(I) \geq l_1(A) + l_2(A) \geq N + (N - d) = 2N - d,$$

and the proof is complete. ∎

Theorem 7.1 follows in two cases.

7.6. COROLLARY. *The Noether complexity of* $X_1(m \times n)$ *is* mn. *The Noether complexity of* $X_2(m \times n)$ *is* $2mn - d$. *The Noether complexity of* $X_t(m \times n)$ (*for* $t \geq 2$) *is at least* $2mn - d$.

Proof. The computation Corollary 5.3 shows that the Noether complexity is at most the asserted values for $t = 1, 2$. The lower bound follows from Proposition 7.5, since parameter matrices for determinantal varieties satisfy M_t. ∎

We now turn to the large rank case, proving Theorem 7.1 for the case $t = m - 1$. We first require a technical lemma, which restricts the partitioning of systems of parameters of determinantal varieties in a strong way.

7.7. LEMMA. *Let* k *and* l *be integers with* $0 \leq k \leq m - t$ *and* $0 \leq l \leq n - t$. *If* $Y_1, \ldots, Y_d$ *are a system of parameters for* $R = (K[\mathbf{X}_{ij}])/(I_{t+1}(m \times n))$, *then at most* $t(k + l)$ *of the parameters can be expressed as linear combinations of the variables* X_{ij} *lying in any* k *rows and* l *columns of* $\mathbf{X}_{ij}$. *In other words, if the ideal*

$$J = \big(X_{ij};\ \text{where } i = i_1, \ldots, i_k \text{ or } j = j_1, \ldots, j_l\big)$$

contains a set of parameters, then the cardinality of that set is at most $t(k + l)$. (*If* $k = 0$, *the notation for* J *is correctly interpreted by dropping the phrase "where* $i = i_1, \ldots, i_k$ *or.*")

Proof. Let B be the submatrix formed by omitting rows $i_1, \ldots, i_k$ and columns $j_1, \ldots, j_l$ from the matrix $A = (X_{ij})$. With J as above, note that $I_{t+1}(A) \subset J + I_{t+1}(B)$. If $Y_1, \ldots, Y_r \subset J$, then $I_{t+1}(A) + (Y_1, \ldots, Y_r) \subset J + I_{t+1}(B)$. Consider the natural surjection

$$\frac{K[X_{ij}]}{I_{t+1}(A) + (Y_1, \ldots, Y_r)} \twoheadrightarrow \frac{K[X_{ij}]}{J + I_{t+1}(B)} \cong \frac{K[B]}{I_{t+1}(B)}.$$

The dimension of the source ring is at least as large as the dimension of the target ring. Computing each, we see that

$$d - r = mt + nt - t^2 \geq (m - k)t + (n - l)t - t^2$$

and we conclude that $r \leq t(k + l)$. ∎

7.8. COROLLARY. *No system of parameters for the determinantal ring* $R = (K[\mathbf{X}_{ij}])/(I_{t+1}(m \times n))$ *can contain* $t + 1$ *parameters from any one row or column. In particular, no system of parameters can contain more than* mt *singletons.*

Proof. This is the case $l = 0$, $k = 1$ (or $k = 0$, $l = 1$) from Lemma 7.7. There would be $t + 1 > t(k + l)$ parameters involving variables from just one row (or column). Furthermore, if there are more than mt parameters of the form X_{ij} (singletons), then at least one of the m rows must contribute more than t of them, a contradiction. ∎

This can be used to derive some useful bounds on the Noether complexity of determinantal rings, which yields the exact lower bound on the Noether complexity in the maximal minor case.

7.9. PROPOSITION. *The Noether complexity of* $X_t(m \times n)$ *is bounded below by*

$$2nt + mt - 2t^2.$$

Proof. Let k denote the number of singletons appearing in a system of parameters of minimal complexity for $(k[\mathbf{X}_{ij}])/(I_{t+1}(m \times n))$. Then the number of elements which are not singletons is $d - k$ and, therefore, the Noether complexity is

$$\mathrm{NC}(I_{t+1}) \geq k + 2(d - k) = 2d - k \geq 2d - mt = 2nt - mt - 2t^2,$$

as needed. ∎

7.10. COROLLARY. *The Noether complexity of* $X_{m-1}(m \times n)$ *of* $m \times n$ *matrices of non-full rank is*

$$(m - 1)(2n - m + 2) = (m - 1)(mn) - (m - 2)d,$$

where d *denotes the dimension of* $X_{m-1}(m \times n)$.

Proof. Note that the dimension in this case is $d = mn - (n - (m - 1))$. The bound Proposition 7.9 gives us that

$$\mathrm{NC}(I_m(m \times n)) \geq (m - 1)(2n + m - 2(m - 1)),$$

while Proposition 5.3 gives

$$\mathrm{NC}\big(I_m(m \times n)\big) \le (m-1)(mn) - (m-2)\big(mn - (n-m+1)\big).$$

Comparing these bounds we get the desired result. ∎

Combining Corollaries 7.10 and 7.6, the proof of Theorem 7.1 is complete.

We point out a related result, describing the complexity of systems of parameters achieved using a "greedy algorithm."

7.11. PROPOSITION. *The complexity of any system of parameters for* $k[X]/I_{t+1}(X)$ *which contains mt singletons is* $tmn - (t-1)d$. *That is, any linear space of codimension d in* $\mathbb{P}(V)$, *disjoint from* X_t *and contained in a "coordinate plane of codimension mt"* $\mathbb{V}$(*mt of the variables* X_{ij}), *must have cobasis complexity* $tmn - d(t-1)$.

Proof. Fix a system of parameters $Y_1, \dots, Y_d$ for $k[X]/I_{t+1}(X)$ and consider the singletons $Y_k = X_{ij}$ in it. As noted in Corollary 7.8, there are at most mt singletons, no $t+1$ of which are in a single row.

If there are exactly mt singletons, then there are t singletons in each row. All other members of the system of parameters have complexity at least $m-t+1$: otherwise, a nonsingleton f of complexity at most $m-t$ together with the $(m-t)t$ singletons involving the variables from the same rows as f is a set of $(m-t)t+1$ parameters involving only $(m-t)$ rows, violating Lemma 7.7. This makes the complexity of the system at least $mt + (m-t+1)(d-mt) = mnt - d(t-1)$. ∎

7.12. CAUTIONARY EXAMPLES. We now construct an example to show that the systems of parameters constructed in Section 5 are not the sparsest possible for all t, while laying another naive conjecture to rest.

Suppose that S_1 and S_2 are graded rings with fixed coordinates $X_1, \dots, X_{n_1}$ and $X'_1, \dots, X'_{n_2}$, respectively. If $Y_1, \dots, Y_{d_1}$ and $Y'_1, \dots, Y'_{d_2}$ are maximally sparse systems of parameters for S_1 and S_2, respectively, then $Y_1, \dots, Y_{d_1}, Y'_1, \dots, Y'_{d_2}$ is a system of parameters for $S_1 \otimes_K S_2$ with respect to the coordinates $X_1, \dots, X_{n_1}, X'_1, \dots, X'_{n_2}$. The corresponding parameter matrix is quite sparse, and has a nice symmetry that may be useful in practice:

$$\begin{pmatrix} \mathsf{A} & \mathbf{0}(d_1 \times n_2) \\ \mathbf{0}(d_2 \times n_1) \times k) & \mathsf{B} \end{pmatrix},$$

where $\mathbf{0}(a \times b)$ denotes an $a \times b$ matrix of zeroes, A is a maximally sparse parameter matrix for S_1, and B is a maximally sparse parameter matrix for S_2. *However, this need not be a maximally sparse system of parameters for* $S_1 \otimes S_2$. In particular, $\mathrm{NC}(S_1 \otimes S_2) \neq \mathrm{NC}(S_1) + \mathrm{NC}(S_2)$ in general.

We now give an explicit example of this phenomenon and use it to construct a system of parameters for a determinantal variety $X_4(9 \times n)$, where $n \geq 9$, which is sparser than the nice system of parameters constructed in Section 5.

7.13. EXAMPLE. Consider a 4×9 matrix of the form

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ a & b & 0 & c & d & 0 & e & f & 0 \\ 0 & g & h & 0 & j & k & m & 0 & p \end{pmatrix}.$$

One can verify that for generic values of indeterminates $a, b, \dots, p$, this matrix has every 4-minor nonzero. Thus it is a parameter matrix for a ring $R = R(4, 9)$ described in Section 4. The Noether cobasis complexity for R is 24, as we have seen.

On the other hand, the ring $R[X] = R \otimes K[X]$ has Noether cobasis complexity at most 24. To see this, we exhibit a system of parameters of complexity 24:

$$\begin{aligned} &U_1 = X + (X_1 + X_2 + X_3), \\ &U_2 = X - (X_4 + X_5 + X_6), \\ &U_3 = X - (X_7 + X_8 + X_9), \\ &Y_3, \\ &Y_4, \end{aligned}$$

where Y_i is the linear form in $X_1, \dots, X_9$ determined by the ith row of the above matrix. Indeed, the difference $U_1 - U_2$ is Y_2 and the difference $U_1 - U_3$ is Y_1, so the ideal these generate contains all the Y_i. However, then because all the X_i are nilpotent modulo the Y_i, the radical of the ideal generated by U_1, U_2, U_3, Y_3, Y_4 contains X as well, and so $\{U_1, U_2, U_3, Y_3, Y_4\}$ form a system of parameters. On the other hand, the complexity of this system of parameters is 24, not 25.

We now show how to use this to create a system of parameters for a determinantal ring of smaller complexity than would be predicted by

Theorem 7.1. Consider a $9 \times n$ matrix, with $n \gg 0$, and let

$$S = \bigotimes_{k=1}^{9+n-1} R_k$$

as in the proof of Proposition 5.4 above, where

$$R_k \cong \frac{K[\text{variables in } D_k]}{(\text{all square-free degree 5 monomials})}$$

is the ring studied in Section 4 (or for $k \leq t$, or $k \geq mn - 1 - t$, it is the polynomial ring in k variables).

The product $R_1 \otimes R_9$ has a system of parameters of complexity 24, by Example 7.13. The remaining R_k have systems of parameters as determined in Section 4. The union of these elements is a system of parameters for S. The total complexity of the union of these parameters is $36n - 3(4n + 20) - 1 = tmn - (t - 1)d - 1$, which is one less than expected. Obviously, if $n \gg 9$, we can make the complexity drop significantly by regrouping several sets of variables in the ring S.

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